

CLASSICAL AND NONCLASSICAL RANDOMNESS IN QUANTUM MEASUREMENTS

DOUGLAS FARENICK, SARAH PLOSKER, AND JERROD SMITH

ABSTRACT. The space $\text{POVM}_{\mathcal{H}}(X)$ of positive operator-valued probability measures on the Borel sets of a compact (or even locally compact) Hausdorff space X with values in $\mathcal{B}(\mathcal{H})$, the algebra of linear operators acting on a d -dimensional Hilbert space \mathcal{H} , is studied from the perspectives of classical and non-classical convexity through a transform Γ that associates any positive operator-valued measure ν with a certain completely positive linear map $\Gamma(\nu)$ of the homogeneous C^* -algebra $C(X) \otimes \mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$. This association is achieved by using an operator-valued integral in which non-classical random variables (that is, operator-valued functions) are integrated with respect to positive operator-valued measures and which has the feature that the integral of a random quantum effect is itself a quantum effect. A left inverse Ω for Γ yields an integral representation, along the lines of the classical Riesz Representation Theorem for linear functionals on $C(X)$, of certain (but not all) unital completely positive linear maps $\phi : C(X) \otimes \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$. The extremal and C^* -extremal points of $\text{POVM}_{\mathcal{H}}(X)$ are determined.

INTRODUCTION

The present paper is a mathematical contribution to quantum probability theory in the setting of finite factors of type I_d , the results of which can be understood from the perspectives of non-relativistic quantum mechanics and quantum information theory.

A measurement of a quantum system is represented, mathematically, by a positive operator-valued probability measure (POVM) ν defined on a σ -algebra $\mathcal{O}(X)$ of measurement events such that whenever a measurement is made with the system in state ρ , the measurement event $E \in \mathcal{O}(X)$ will occur with probability $\text{Tr}(\rho\nu(E))$ [5]. In this formulation, X is a locally compact Hausdorff space of measurement outcomes, $\mathcal{O}(X)$ is a σ -algebra of Borel sets of X , ρ is a density operator acting on a separable Hilbert space \mathcal{H} , and $\text{Tr}(\cdot)$ is the canonical trace on the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators acting on \mathcal{H} . In practice, quantum measurements of an actual physical system are made by way of some apparatus and in such cases the sample space X is typically assumed to be finite. Consequently, a great deal of the literature on the mathematical aspects of POVMs deals only with finite sample spaces X .

On the other hand, probability theory and its use in physics does not require the sample spaces to be finite. Moreover, in theory, a POVM defined on an arbitrary (perhaps infinite) sample space X corresponds to a physically realisable quantum measurement. Therefore, one of our primary goals is to approach the theory of quantum measurement under the assumptions that X be arbitrary. More precisely, we consider the fairly general situation in which the sample space X is a compact Hausdorff topological space and the Hilbert space \mathcal{H} has finite dimension d . While

this level of abstraction is in accordance with the generalities present in the axioms for measurements of quantum systems, it is also useful in the mathematical analysis and interpretation of measurements of quantum systems with finitely many outcomes. It is not difficult to modify our work herein so that the assumption of the compactness of X be weakened to the requirement that X be locally compact, but the extension of our work from d -dimensional Hilbert space to infinite-dimensional Hilbert space is of a very different nature, which we do not intend to address herein.

For a fixed sample space X , the set of all measuring apparaata of a quantum system \mathcal{H} is denoted by $\text{POVM}_{\mathcal{H}}(X)$. This is a convex set in which a (classical) convex combination of POVMs corresponds to a random choice of measuring apparatus. However, $\text{POVM}_{\mathcal{H}}(X)$ exhibits a stronger, nonclassical convexity property, namely that of C^* -convexity, and we herein consider $\text{POVM}_{\mathcal{H}}(X)$ from both the classical and nonclassical geometric points of view. In C^* -convexity, scalar-valued convex coefficients are replaced by operator-valued convex coefficients. Thus, a C^* -convex combination of POVMs corresponds to a nonclassical (or quantum) random choice of apparatus. A set of C^* -convex coefficients corresponds to the noise (or Kraus) operators of a unital quantum channel. Thus, if a measurement ν is obtained through a proper C^* -convex combination of measurements ν_1, \dots, ν_n , then ν is a coarser measurement than each of the ν_j . Conceptually, a sharp measurement ν is one in which the only measurements that are coarser than ν are those that are (unitarily) equivalent to ν . We shall show in Theorem 5.1 that this conceptual notion of sharpness coincides with the common notion of sharpness, namely that of a classical observable (or, in POVM terminology, a projection-valued measure) [21].

A further nonclassical development made herein is analytical. By taking the word “measure” in its literal sense, we show that for every $\nu \in \text{POVM}_{\mathcal{H}}(X)$ one may define an operator-valued integral $\int_X \psi d\nu$ for all quantum random variables ψ . This integral has the feature that the integral of a function whose values are quantum effects is again a quantum effect. Through the identification of the C^* -algebra $C(X) \otimes \mathcal{B}(\mathcal{H})$ with the unital homogeneous C^* -algebra of all continuous functions $f : X \rightarrow \mathcal{B}(\mathcal{H})$, we show that the map $f \mapsto \int_X f d\nu$ is a unital completely positive (ucp) linear map $\phi_{\nu} : C(X) \otimes \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$.

The representation of $\nu \in \text{POVM}_{\mathcal{H}}(X)$ by a ucp map ϕ_{ν} is a function, which we call the Γ -transform, by which the space $\text{POVM}_{\mathcal{H}}(X)$ can be studied using the theory of completely positive linear maps [23]. Because of the noncommutativity of operator algebra, Γ is far from being surjective and does not preserve the affine structure of $\text{POVM}_{\mathcal{H}}(X)$; but Γ does have a left inverse Ω which is C^* -affine. We endow $\text{POVM}_{\mathcal{H}}(X)$ with a natural topology in which $\text{POVM}_{\mathcal{H}}(X)$ is a compact space. Therefore, all quantum measurements are approximated by convex combinations of extremal quantum measurements.

We analyse the extremal POVMs, using some earlier work in [9, 22] for the case of finite sample spaces, to obtain a nonclassical version of the classical theorem in probability that asserts that a probability measure is extremal if and only if its mass is concentrated at a point of the sample space. This result (Theorem 4.1) can be found in the relatively recent literature [7, 8]; however, the approach we take here is rather different and adheres to our over-arching theme of using “quantum methods” for proving statements about quantum probabilities. Using nonclassical convexity [14, 16], we prove that every quantum measurement with

finitely many outcomes is a coarsening of sharp measurements, and that every quantum measurement with arbitrary outcomes is statistically approximated by a coarsening of sharp measurements with finitely many outcomes.

1. NOTATION, TERMINOLOGY, AND ASSUMPTIONS

General references for POVMs and completely positive linear maps are [5, 10, 20] and [23], respectively.

1.1. Assumption. By X we denote a compact Hausdorff space and by \mathcal{H} a d -dimensional Hilbert space. Let $\{e_1, \dots, e_d\}$ be an orthonormal basis for \mathcal{H} , which henceforth is assumed to be fixed. For each $i, j \in \{1, \dots, d\}$, we denote by $e_{ij} \in \mathcal{B}(\mathcal{H})$ the unique operator that sends e_j to e_i and all other e_k ($k \neq j$) to 0.

1.2. States, effects, automorphisms, and POVMs. The real vector space of all selfadjoint operators acting on \mathcal{H} is denoted by $\mathcal{B}(\mathcal{H})_{\text{sa}}$ and $\mathcal{B}(\mathcal{H})_+ \subset \mathcal{B}(\mathcal{H})_{\text{sa}}$ denotes the cone of positive operators. The state space of \mathcal{H} is denoted by $S(\mathcal{H})$ and consists of all $\rho \in \mathcal{B}(\mathcal{H})_+$ of trace $\text{Tr } \rho = 1$. In particular, $e \in S(\mathcal{H})$ denotes $\frac{1}{d}1$, where $1 \in \mathcal{B}(\mathcal{H})$ is the identity operator. A quantum effect is a positive operator $h \in \mathcal{B}(\mathcal{H})_+$ with the property that $0 \leq \lambda \leq 1$ for every eigenvalue λ of h . The set of quantum effects is denoted by $\text{Eff}(\mathcal{H})$. By $\text{Aut}(\mathcal{B}(\mathcal{H}))$ we denote the automorphism group for the C*-algebra $\mathcal{B}(\mathcal{H})$. Thus, $\alpha \in \text{Aut}(\mathcal{B}(\mathcal{H}))$ if and only if there is a unitary operator $u \in \mathcal{B}(\mathcal{H})$ such that $\alpha(x) = u^*xu$ for every $x \in \mathcal{B}(\mathcal{H})$. The σ -algebra of Borel sets of X is denoted by $\mathcal{O}(X)$.

Definition 1.1. A function $\nu : \mathcal{O}(X) \rightarrow \mathcal{B}(\mathcal{H})$ is a *positive operator valued probability measure* (POVM) on X if:

- (1) $\nu(E) \in \text{Eff}(\mathcal{H})$, for every $E \in \mathcal{O}(X)$;
- (2) for every countable collection $\{E_k\}_{k \in \mathbb{N}} \subset \mathcal{O}(X)$ with $E_k \cap E_{k'} = \emptyset$ for $k' \neq k$,

$$\nu\left(\bigcup_{k \in \mathbb{N}} E_k\right) = \sum_{k \in \mathbb{N}} \nu(E_k),$$

where the convergence of the right hand side of the equality above is with respect to the weak operator topology of $\mathcal{B}(\mathcal{H})$;

- (3) $\nu(X) = 1 \in \mathcal{B}(\mathcal{H})$.

If, in addition, $\nu(E)$ is a projection for every $E \in \mathcal{O}(X)$, then ν is said to be *sharp* (or *classical*).

In light of the correspondence between physical quantum measurements and mathematical POVMs, we shall frequent use the terms *quantum measurement* or *quantum instrument* for elements of $\text{POVM}_{\mathcal{H}}(X)$.

The set of all positive operator valued measures on X with values in $\mathcal{B}(\mathcal{H})$ is denoted by $\text{POVM}_{\mathcal{H}}(X)$. We shall identify the space $P(X)$ of Borel probability measures with the subset $\{\mu \cdot 1 : \mu \in P(X)\} \subset \text{POVM}_{\mathcal{H}}(X)$ so that we think of ordinary probability measures as scalar-valued POVMs.

Definition 1.2. The *support* of $\nu \in \text{POVM}_{\mathcal{H}}(X)$ is the smallest closed subset $K_{\nu} \subset X$ for which $\nu(X \setminus K_{\nu}) = 0$.

If the support of $\nu \in \text{POVM}_{\mathcal{H}}(X)$ consists of a single point, say $K_{\nu} = \{x_0\}$, then ν is a *Dirac measure* and is necessarily of the form $\nu = \delta_{x_0}1$, where $\delta_{x_0} \in P(X)$ satisfies, for $E \in \mathcal{O}(X)$, $\delta_{x_0}(E) = 1$ if $x_0 \in E$ and $\delta_{x_0}(E) = 0$ otherwise.

If $\nu \in \text{POVM}_{\mathcal{H}}(X)$ has finite support $K_{\nu} = \{x_1, \dots, x_m\}$, then each $h_j = \nu(\{x_j\}) \neq 0$ and

$$\nu = \sum_{j=1}^m \delta_{x_j} h_j.$$

1.3. Completely positive linear maps of homogeneous C*-algebras. If X is compact, then the set $C(X)$ of all continuous functions $X \rightarrow \mathbb{C}$ is a unital, abelian C*-algebra, and the C*-algebra $C(X) \otimes \mathcal{B}(\mathcal{H})$ is naturally identified with the homogeneous C*-algebra of all continuous functions $f : X \rightarrow \mathcal{B}(\mathcal{H})$.

A linear map $\phi : C(X) \otimes \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is completely positive if

$$\phi \otimes \text{id}_{\mathcal{M}_p} : C(X) \otimes \mathcal{B}(\mathcal{H}) \otimes \mathcal{M}_p \rightarrow \mathcal{B}(\mathcal{H}) \otimes \mathcal{M}_p$$

preserves positivity for every algebra \mathcal{M}_p of complex $p \times p$ matrices. A completely positive linear map that preserves the identity is called a ucp map (unital, completely positive). Let

$$\text{UCP}_{\mathcal{H}}(X) = \{\phi : \phi \text{ is a ucp map } C(X) \otimes \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})\}.$$

Definition 1.3. A ucp map $\varrho_{x_0} \in \text{UCP}_{\mathcal{H}}(X)$ of the form

$$\varrho_{x_0}(f) = f(x_0), \quad f \in C(X) \otimes \mathcal{B}(\mathcal{H}),$$

for some fixed $x_0 \in X$, is said to be *spectral*.

The spectral ucp maps coincide with the ucp maps $\varrho \in \text{UCP}_{\mathcal{H}}(X)$ that have the property $\varrho(fg) = \varrho(f)\varrho(g)$ for all $f, g \in C(X) \otimes \mathcal{B}(\mathcal{H})$. Any two spectral ucp maps $\varrho_{x_0}, \varrho_{x_1} \in \text{UCP}_{\mathcal{H}}(X)$ are unitarily equivalent—that is, $\varrho_{x_1} = \alpha \circ \varrho_{x_0}$ for some automorphism $\alpha \in \text{Aut}(\mathcal{B}(\mathcal{H}))$ —if and only if $x_1 = x_0$. Thus, the set

$$\text{Sp}_{\mathcal{H}}(X) = \{\varrho_{x_0} : x_0 \in X\}$$

can be identified with the spectrum [11] of the C*-algebra $C(X) \otimes \mathcal{B}(\mathcal{H})$.

Definition 1.4. A ucp map $\phi \in \text{UCP}_{\mathcal{H}}(X)$ of the form

$$\phi(f) = \sum_{j=1}^m t_j^* f(x_j) t_j, \quad f \in C(X) \otimes \mathcal{B}(\mathcal{H}),$$

for some $x_1, \dots, x_m \in X$ (not necessarily distinct) and $t_1, \dots, t_m \in \mathcal{B}(\mathcal{H})$ is said to be *elementary*.

The set of all elementary maps $C(X) \otimes \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is denoted by $\text{E}_{\mathcal{H}}(X)$, and the set of all elementary maps $f \mapsto \sum_{j=1}^m t_j^* f(x_j) t_j$ in which each t_j is a positive operator is denoted by $\text{E}_{\mathcal{H}}^+(X)$.

Thus, we have a heirarchy:

$$\text{Sp}_{\mathcal{H}}(X) \subset \text{E}_{\mathcal{H}}^+(X) \subset \text{E}_{\mathcal{H}}(X) \subset \text{UCP}_{\mathcal{H}}(X).$$

1.4. Convexity and C*-Convexity. The sets $\text{POVM}_{\mathcal{H}}(X)$ and $\text{UCP}_{\mathcal{H}}(X)$ are not only convex, but they are also C*-convex. To explain the nonclassical notion of C*-convexity, we consider an abstract context that will capture the nonclassical convexity of both $\text{POVM}_{\mathcal{H}}(X)$ and $\text{UCP}_{\mathcal{H}}(X)$.

Let Y be a nonempty set and assume that $V_{\mathcal{H}}(Y)$ is the vector space (under pointwise addition and scalar multiplication) of all functions $\zeta : Y \rightarrow \mathcal{B}(\mathcal{H})$. If $z_1, z_2 \in \mathcal{B}(\mathcal{H})$ and $\zeta \in V_{\mathcal{H}}(Y)$, then define

$$z_1 \zeta z_2 : Y \rightarrow \mathcal{B}(\mathcal{H}) \quad \text{given by} \quad y \mapsto z_1 \zeta(y) z_2 \in \mathcal{B}(\mathcal{H}).$$

This left and right multiplication of elements of V by elements of $\mathcal{B}(\mathcal{H})$ gives V the algebraic structure of a $\mathcal{B}(\mathcal{H})$ -bimodule.

In particular, one may apply unitary similarity transformations to functions $\zeta : Y \rightarrow \mathcal{B}(\mathcal{H})$.

Definition 1.5. Two functions $\zeta, \zeta' \in V_{\mathcal{H}}(Y)$ are *unitarily equivalent* if $\zeta' = u^* \zeta u$ for some unitary operator $u \in \mathcal{B}(\mathcal{H})$.

Definition 1.6. Assume that $a_1, \dots, a_m \in \mathcal{B}(\mathcal{H})$ and $\zeta_1, \dots, \zeta_m \in V_{\mathcal{H}}(Y)$. Then:

- (1) $a_1, \dots, a_m \in \mathcal{B}(\mathcal{H})$ are called *C*-convex coefficients* if $\sum_{j=1}^m a_j^* a_j = 1$;
- (2) a *C*-convex combination* of ζ_1, \dots, ζ_m is a function $\zeta : Y \rightarrow \mathcal{B}(\mathcal{H})$ of the form $\zeta = \sum_{j=1}^m a_j^* \zeta_j a_j$, where $a_1, \dots, a_m \in \mathcal{B}(\mathcal{H})$ are C*-convex coefficients;
- (3) a *proper C*-convex combination* of ζ_1, \dots, ζ_m is a function $\zeta : Y \rightarrow \mathcal{B}(\mathcal{H})$ of the form $\zeta = \sum_{j=1}^m a_j^* \zeta_j a_j$, where $a_1, \dots, a_m \in \mathcal{B}(\mathcal{H})$ are invertible C*-convex coefficients.

Furthermore, a subset $K \subset V_{\mathcal{H}}(Y)$ is *C*-convex* over $\mathcal{B}(\mathcal{H})$ if K contains all C*-convex combinations of its elements.

The C*-convex sets of interest here are:

- (1) $\text{POVM}_{\mathcal{H}}(X)$, using $Y = \mathcal{O}(X)$ in the definitions above;
- (2) $\text{UCP}_{\mathcal{H}}(X)$, using $Y = X$; and
- (3) $E_{\mathcal{H}}(X)$, a C*-convex subset of $\text{UCP}_{\mathcal{H}}(X)$.

Definition 1.7. If $R \subset V_{\mathcal{H}}(Y)$ is a nonempty subset, then

- (1) the *C*-convex hull* of R is the set $C^*\text{conv } R$ consisting of all elements of $V_{\mathcal{H}}(Y)$ attained from all possible C*-convex combinations of elements of R , and
- (2) the *proper C*-convex hull* of R is the set $C_p^*\text{conv } R$ consisting of all elements of $V_{\mathcal{H}}(Y)$ attained from all possible proper C*-convex combinations of elements of R .

Observe that $C^*\text{conv } R$ is itself a C*-convex set and that any C*-convex set is also a convex set (in the classical sense). With this formalism, we have

$$\text{Sp}_{\mathcal{H}}(X) \subset E_{\mathcal{H}}^+(X) \subset E_{\mathcal{H}}(X) = C^*\text{-conv}(\text{Sp}_{\mathcal{H}}(X)) \subset \text{UCP}_{\mathcal{H}}(X).$$

If K is a C*-convex subset of $V_{\mathcal{H}}(Y)$ and if $\zeta \in K$, then $u^* \zeta u \in K$ for every unitary operator $u \in \mathcal{B}(\mathcal{H})$. That is, K is closed under unitary similarities. Furthermore, it is easy to show that every $\zeta \in K$ is a proper C*-convex combination of elements unitarily equivalent to it. The C*-extremal elements ζ of K are the ones in which this is the only way to represent ζ as a proper C*-convex combination of other elements of K .

Definition 1.8. An element ζ in a C^* -convex subset $K \subset V_{\mathcal{H}}(Y)$ is a *C^* -extreme point* if the only manner in which to express ζ as a proper C^* -convex combination of $\zeta_1, \dots, \zeta_m \in K$ is by way of ζ_j of the form $\zeta_j = u_j^* \zeta u_j$ for some unitary operators $u_1, \dots, u_m \in \mathcal{B}(\mathcal{H})$.

Let the sets of extreme points and C^* -extreme points of a C^* -convex set $K \subset V_{\mathcal{H}}(Y)$ be denoted, respectively, by

$$\text{ext } K \quad \text{and} \quad C^*\text{ext } K.$$

We have the following relationship between the two sets.

Proposition 1.1. *If $K \subset V_{\mathcal{H}}(Y)$ is C^* -convex, then $C^*\text{ext } K \subset \text{ext } K$.*

Proof. Assume that $\zeta \in K$ is a C^* -extreme point and that $\zeta = \lambda \zeta_1 + (1 - \lambda) \zeta_2$ for some $\zeta_1, \zeta_2 \in K$ and some real number $\lambda \in (0, 1)$. Set $a_1 = \sqrt{\lambda} \mathbf{1}$ and $a_2 = \sqrt{1 - \lambda} \mathbf{1}$ to obtain the proper C^* -convex combination $\zeta = \sum_{j=1}^2 a_j^* \zeta_j a_j$. By hypothesis, $\zeta_j = u_j^* \zeta u_j$ for some unitaries $u_1, u_2 \in \mathcal{B}(\mathcal{H})$. Fix $y_0 \in Y$ so that $\zeta(y_0) = \lambda u_1^* \zeta(y_0) u_1 + (1 - \lambda) u_2^* \zeta(y_0) u_2$. Because \mathcal{H} has finite dimension, the operator algebra $\mathcal{B}(\mathcal{H})$ is a Hilbert space with respect to the Hilbert-Schmidt norm $\|\cdot\|_2$. Hence, the equation $\zeta(y_0) = \lambda u_1^* \zeta(y_0) u_1 + (1 - \lambda) u_2^* \zeta(y_0) u_2$ represents the vector $\zeta(y_0)$, which lies on the sphere of radius $\|\zeta(y_0)\|_2$, as a convex combination of the vectors $u_1^* \zeta(y_0) u_1$ and $u_2^* \zeta(y_0) u_2$, which also lie on the same sphere. Because the sphere of a Hilbert space contains no nontrivial line segments, the vectors $u_1^* \zeta(y_0) u_1$ and $u_2^* \zeta(y_0) u_2$ must coincide with $\zeta(y_0)$. As this is true for every y_0 , we obtain $\zeta_1 = \zeta_2 = \zeta$, and so ζ is an extreme point of K . \square

Returning now to $\text{POVM}_{\mathcal{H}}(X)$ and $\text{UCP}_{\mathcal{H}}(X)$, which are the C^* -convex sets of interest to us here, we summarise below the current state of knowledge regarding the extreme and C^* -extreme points of these sets.

- (1) For arbitrary X , the extreme points of $\text{UCP}_{\mathcal{H}}(X)$ are deduced from a quite general theorem of Arveson [2, Theorem 1.4.2].
- (2) For arbitrary X , the C^* -extreme points of $\text{UCP}_{\mathcal{H}}(X)$ can be deduced from a theorem of Farenick and Zhou [16, Theorem 2.1].
- (3) For finite X , the extreme points of $\text{POVM}_{\mathcal{H}}(X)$ have been determined by Parthasarathy [22] and by D'Ariano, Lo Presti, and Perinotti [9], and for arbitrary X the extreme points of $\text{POVM}_{\mathcal{H}}(X)$ are characterised by Chiribella, D'Ariano, and Schlingemann in [7, 8].
- (4) For arbitrary X , the extreme points and C^* -extreme points of $\text{POVM}_{\mathcal{H}}(X)$ are determined in Theorems 4.1 and 5.1 of the present paper.

2. QUANTUM RANDOM VARIABLES AND INTEGRATION

Definition 2.1. A *quantum random variable* is a function $\psi : X \rightarrow \mathcal{B}(\mathcal{H})$ that is *Borel measurable* in the sense that the complex-valued functions

$$x \mapsto \text{Tr}(\rho \psi(x))$$

are Borel measurable for every state $\rho \in \mathcal{S}(\mathcal{H})$.

Equivalently, $\psi : X \rightarrow \mathcal{B}(\mathcal{H})$ is Borel measurable if, for every pair of vectors $\xi, \eta \in \mathcal{H}$, the complex-valued function $x \mapsto \langle \psi(x) \xi, \eta \rangle$ is Borel measurable. Our aim in this section is to define, using the procedure set out in [15], a positive-preserving

operator-valued integral $\int_X \psi d\nu$ for any Borel measurable function $\psi : X \rightarrow \mathcal{B}(\mathcal{H})$ and any $\nu \in \text{POVM}_{\mathcal{H}}(X)$.

Every positive operator $h \in \mathcal{B}(\mathcal{H})$ has a unique positive square root $h^{1/2}$. Thus, if $\psi : X \rightarrow \mathcal{B}(\mathcal{H})$ is a function for which $\psi(x)$ is a positive operator for every $x \in X$, then $\psi^{1/2} : X \rightarrow \mathcal{B}(\mathcal{H})$ denotes the function $\psi^{1/2}(x) = (\psi(x))^{1/2}$.

The following observation will be useful.

Proposition 2.1. *If $\psi : X \rightarrow \mathcal{B}(\mathcal{H})$ is a positive quantum random variable, then $\psi^{1/2}$ is a (positive) quantum random variable.*

Proof. Assume first that $\psi(x)$ is positive and invertible for every $x \in X$. Because sums and products of scalar-valued measurable functions are measurable, if one invokes an iterative procedure to compute $\psi(x)^{1/2}$ —such as the one in [19, Algorithm 2], which is a Newton-type iteration combined with a Cholesky factorisation—then for each state $\rho \in \mathcal{S}(\mathcal{H})$ the function

$$x \mapsto \text{Tr}(\rho\psi(x)^{1/2})$$

is a pointwise limit of a sequence of measurable functions. Thus, $\psi^{1/2}$ is a quantum random variable. In the case where $\psi(x)$ is not invertible for all $x \in X$, then $\psi^{1/2}$ is a pointwise limit of $x \mapsto (\psi(x) + \frac{1}{n}\mathbf{1})^{1/2}$ and, hence, is measurable. \square

2.1. The Principal Radon-Nikodým Derivative.

Definition 2.2. If $\nu_1, \nu_2 \in \text{POVM}_{\mathcal{H}}(X)$, then ν_2 is *absolutely continuous* with respect to ν_1 , denoted by $\nu_2 \ll_{\text{ac}} \nu_1$, if $\nu_2(E) = 0$ for every $E \in \mathcal{O}(X)$ for which $\nu_1(E) = 0$.

If $\nu \in \text{POVM}_{\mathcal{H}}(X)$, then a probability measure μ is obtained from ν via

$$(1) \quad \mu(E) = \frac{\text{Tr}(\nu(E))}{d}, \text{ for every } E \in \mathcal{O}(X).$$

Because the trace functional maps nonzero positive operators to strictly positive real numbers, the measures μ and ν are mutually absolutely continuous: $\mu \ll_{\text{ac}} \nu$ and $\nu \ll_{\text{ac}} \mu$.

Recall that $\{e_1, \dots, e_d\}$ is a fixed orthonormal basis of \mathcal{H} . Because $\nu \ll_{\text{ac}} \mu$, each of the d^2 complex measures $\nu_{ij} : \mathcal{O}(X) \rightarrow \mathbb{C}$, defined by $\nu_{ij}(E) = \langle \nu(E)e_j, e_i \rangle$, has the property that $\nu_{ij} \ll_{\text{ac}} \mu$. Hence, by the (classical) Radon-Nikodým Theorem, there is a unique $\frac{d\nu_{ij}}{d\mu} \in L^1(X, \mu)$ such that

$$\nu_{ij}(E) = \int_E \frac{d\nu_{ij}}{d\mu} d\mu, \text{ for all } E \in \mathcal{O}(X).$$

These scalar Radon-Nikoým derivatives give rise to an operator-valued Borel function $\frac{d\nu}{d\mu} : X \rightarrow \mathcal{B}(\mathcal{H})$ via

$$(2) \quad \frac{d\nu}{d\mu} = \sum_{i,j=1}^d \frac{d\nu_{ij}}{d\mu} \otimes e_{ij}.$$

Notice that for any $\xi = \sum_{k=1}^d \xi_k e_k \in \mathcal{H}$,

$$\left\langle \frac{d\nu}{d\mu}(x)\xi, \xi \right\rangle = \sum_{i,j=1}^d \frac{d\nu_{ij}}{d\mu} \xi_j \overline{\xi_i}.$$

Hence, for all $\xi \in \mathcal{H}$ and $E \in \mathcal{O}(X)$,

$$\int_E \left\langle \frac{d\nu}{d\mu}(x)\xi, \xi \right\rangle d\mu(x) = \sum_{i,j=1}^d \left(\int_E \frac{d\nu_{ij}}{d\mu} d\mu \right) \xi_j \overline{\xi_i} = \langle \nu(E)\xi, \xi \rangle \geq 0.$$

This proves that $\frac{d\nu}{d\mu}(x)$ is a positive operator for μ -almost all $x \in X$; for such x let $\left(\frac{d\nu}{d\mu}(x)\right)^{1/2}$ denote the positive square root (in $\mathcal{B}(\mathcal{H})$) of the positive operator $\frac{d\nu}{d\mu}(x)$. Now define $\left(\frac{d\nu}{d\mu}\right)^{1/2} : X \rightarrow \mathcal{B}(\mathcal{H})$ to be $\left(\frac{d\nu}{d\mu}(x)\right)^{1/2}$ at those $x \in X$ for which $\frac{d\nu}{d\mu}(x)$ is a positive operator, and zero otherwise.

Definition 2.3. If $\nu \in \text{POVM}_{\mathcal{H}}(X)$ and if $\mu \in P(X)$ is the induced classical probability measure defined in (1), then the Borel function $\frac{d\nu}{d\mu}$ defined in (2) is called the *principal Radon-Nikodým derivative of ν* .

Unlike the classical case, whenever $d > 1$ the principal Radon-Nikodým derivative of ν depends on the pre-selected choice of orthonormal basis $\{e_1, \dots, e_d\}$ of \mathcal{H} . If one had chosen a different orthonormal basis, say $\{e'_1, \dots, e'_d\}$, then the resulting principal Radon-Nikodým derivative computed in this new basis is simply that of $\alpha \circ \nu$ in the originally selected basis, where α is the automorphism induced by the unitary operator that transforms the basis $\{e'_1, \dots, e'_d\}$ to the basis $\{e_1, \dots, e_d\}$. The following proposition is even more general.

Recall that a *unital quantum channel* is a linear map $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ such that \mathcal{E} is unital, completely positive, and trace preserving (that is, $\text{Tr} \circ \mathcal{E} = \text{Tr}$). Note that $\mathcal{E} \circ \nu \in \text{POVM}_{\mathcal{H}}(X)$ for every $\nu \in \text{POVM}_{\mathcal{H}}(X)$.

Proposition 2.2. Assume that $\nu \in \text{POVM}_{\mathcal{H}}(X)$ and that $\mathcal{E} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is a unital quantum channel. Let μ^ν and $\mu^{\mathcal{E} \circ \nu}$ be the probability measures induced by ν and $\mathcal{E} \circ \nu$ in accordance with (1). Then there is a $\mu \in P(X)$ such that

- (1) $\mu = \mu^\nu = \mu^{\mathcal{E} \circ \nu}$ and
- (2) $\frac{d(\mathcal{E} \circ \nu)}{d\mu} = \mathcal{E} \circ \frac{d\nu}{d\mu}$.

Proof. The channel \mathcal{E} is trace preserving, so for any $E \in \mathcal{O}(X)$

$$\mu^{\mathcal{E} \circ \nu}(E) = \frac{1}{d} \text{Tr}(\mathcal{E}(\nu(E))) = \frac{1}{d} \text{Tr}(\nu(E)) = \mu^\nu(E).$$

The desired measure is $\mu = \mu^{\mathcal{E} \circ \nu} = \mu^\nu$.

Let $a = \sum_{i,j=1}^d \alpha_{ij} e_{ij} \in \mathcal{B}(\mathcal{H})$ and consider $a^* \nu a$. If $\mu(E) = 0$, then $\nu(E) = 0$ and $a^* \nu(E) a = 0$; therefore $a^* \nu a \ll_{\text{ac}} \mu$. Fix i, j and consider the (i, j) -coordinate

measure of $a^* \nu a$:

$$\omega_{ij} = \sum_{l=1}^d \sum_{k=1}^d \overline{\alpha_{li}} \alpha_{kj} \nu_{lk}.$$

Since $a^* \nu a \ll_{\text{ac}} \mu$, we have $\omega_{ij} \ll_{\text{ac}} \mu$ and so we may consider the Radon-Nikodým derivative

$$\frac{d\omega_{ij}}{d\mu} = \sum_{l=1}^d \sum_{k=1}^d \overline{\alpha_{li}} \alpha_{kj} \left(\frac{d\nu_{lk}}{d\mu} \right) = \left(a^* \frac{d\nu}{d\mu} a \right)_{ij}.$$

Therefore, $\frac{d(a^* \nu a)}{d\mu} = a^* \frac{d\nu}{d\mu} a$.

Consider the Kraus decomposition of the channel \mathcal{E} :

$$\mathcal{E}(y) = \sum_{j=1}^q a_j^* y a_j, \quad y \in \mathcal{B}(\mathcal{H}), \quad \text{where } \sum_{j=1}^q a_j^* a_j = \sum_{j=1}^q a_j a_j^* = 1.$$

By linearity of the scalar Radon-Nikodým derivative, $\frac{d(\mathcal{E} \circ \nu)}{d\mu} = \mathcal{E} \circ \frac{d\nu}{d\mu}$. \square

2.2. Integrable Functions. If $f, \psi : X \rightarrow \mathcal{B}(\mathcal{H})$ are quantum random variables such that $\psi(x) \in \mathcal{B}(\mathcal{H})_+$ for all $x \in X$, then $\psi^{1/2}$ is measurable (Proposition 2.1) and, thus, the function $\psi^{1/2} f \psi^{1/2}$ is Borel measurable.

Definition 2.4. Assume that $\nu \in \text{POVM}_{\mathcal{H}}(X)$ and that $\frac{d\nu}{d\mu}$ is the principal Radon-Nikodým derivative of ν .

- (1) If $f : X \rightarrow \mathcal{B}(\mathcal{H})$ is a Borel function, then f is said to be ν -integrable if, for every state $\rho \in \mathcal{S}(\mathcal{H})$, the complex-valued function

$$f_\rho(x) = \text{Tr} \left(\rho \left(\frac{d\nu}{d\mu}(x) \right)^{1/2} f(x) \left(\frac{d\nu}{d\mu}(x) \right)^{1/2} \right), \quad x \in X,$$

is μ -integrable.

- (2) The integral of a ν -integrable function $f : X \rightarrow \mathcal{B}(\mathcal{H})$ is defined to be the unique operator acting on \mathcal{H} having the property that

$$\text{Tr} \left(\rho \int_X f d\nu \right) = \int_X f_\rho d\mu$$

for every state ρ of \mathcal{H} .

Example 2.3. The integral of an effect-valued function is an effect.

To verify this claim, choose $\nu \in \text{POVM}_{\mathcal{H}}(X)$ and let μ be its principal Radon-Nikodým derivative. Assume that $f : X \rightarrow \text{Eff}(\mathcal{H})$ is ν -integrable. Because $0 \leq f(x) \leq 1$ in $\mathcal{B}(\mathcal{H})_{\text{sa}}$, for every state $\rho \in \mathcal{S}(\mathcal{H})$ we have

$$0 \leq \rho^{1/2} \left(\frac{d\nu}{d\mu}(x) \right)^{1/2} f(x) \left(\frac{d\nu}{d\mu}(x) \right)^{1/2} \rho^{1/2} \leq \rho^{1/2} \left(\frac{d\nu}{d\mu}(x) \right) \rho^{1/2}$$

for μ -almost all $x \in X$. Thus, for every $\rho \in \mathcal{S}(\mathcal{H})$, we have

$$\int_X f_\rho d\mu \leq \int_X \text{Tr} \left(\rho \frac{d\nu}{d\mu} \right) d\mu$$

and so $0 \leq \int_X f d\nu \leq \int_X \left(\frac{d\nu}{d\mu} \right) d\nu = \nu(X) = 1 \in \mathcal{B}(\mathcal{H})$. \diamond

Example 2.4. *The principal Radon-Nikodým derivative of $\nu = \sum_{j=1}^n \delta_{x_j} h_j$ and the corresponding integral formula.*

Here, we assume that $h_1, \dots, h_n \in \mathcal{B}(\mathcal{H})_+$ are nonzero and satisfy $\sum_j h_j = 1$ and that $\{x_1, \dots, x_n\}$ is a set of n distinct points of X . The measurement $\nu \in \text{POVM}_{\mathcal{H}}(X)$ is defined by

$$\nu(E) = \sum_{j=1}^n \delta_{x_j}(E) h_j, \quad E \in \mathcal{O}(X).$$

If χ_E denotes the characteristic (or indicator) function of any measurement event $E \in \mathcal{O}(X)$, then

$$\frac{d\nu}{d\mu} = \sum_{j=1}^n \left(\frac{d}{\text{Tr}(h_j)} \chi_{\{x_j\}} \right) h_j$$

and

$$\int_X f d\nu = \sum_{j=1}^n h_j^{1/2} f(x_j) h_j^{1/2},$$

for every Borel function $f : X \rightarrow \mathcal{B}(\mathcal{H})$. \diamondsuit

2.3. Quantum Integration is a Completely Positive Operator. The following theorem is the first main result of the present paper. To set the notation used in the proof, for any operator algebra \mathcal{A} we let $\mathcal{M}_d(\mathcal{A})$ denote the C^* -algebra of $d \times d$ matrices with entries from \mathcal{A} . An element $F \in \mathcal{M}_d(\mathcal{A})$ is a matrix $F = [f_{k\ell}]_{k,\ell=1}^d$ of elements $f_{k\ell} \in \mathcal{A}$.

Theorem 2.5. *If $\nu \in \text{POVM}_{\mathcal{H}}(X)$, then there is a unital completely positive linear map*

$$\phi_\nu : C(X) \otimes \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$$

such that

$$\phi_\nu(f) = \int_X f d\nu,$$

for every $f \in C(X) \otimes \mathcal{B}(\mathcal{H})$.

Proof. Choose $\nu \in \text{POVM}_{\mathcal{H}}(X)$ and let μ be its principal Radon-Nikodým derivative. Because \mathcal{H} has finite dimension d , to prove that ϕ_ν is completely positive it is sufficient to show that the linear function $s_{\phi_\nu} : \mathcal{M}_d(C(X) \otimes \mathcal{B}(\mathcal{H})) \rightarrow \mathbb{C}$ defined by

$$s_{\phi_\nu}([f_{k\ell}]_{k,\ell}) = \frac{1}{d} \sum_{k,\ell=1}^d \langle \phi_\nu(f_{k\ell}) e_\ell, e_k \rangle$$

maps positive elements of $\mathcal{M}_d(C(X) \otimes \mathcal{B}(\mathcal{H}))$ to nonnegative real numbers [23, Theorem 6.1]. Note that $G = [g_{k\ell}]_{k,\ell} \in \mathcal{M}_d(C(X) \otimes \mathcal{B}(\mathcal{H}))$ is positive if, for every $x \in X$, the operator

$$G(x) = [g_{k\ell}(x)]_{k,\ell} \in \mathcal{M}_d(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^d) = \mathcal{B}\left(\bigoplus_1^d \mathcal{H}\right)$$

is positive.

Thus, let $F = [f_{k\ell}]_{k,\ell} \in \mathcal{M}_d(C(X) \otimes \mathcal{B}(\mathcal{H}))$ be positive. The principal Radon-Nikodým derivative is positive μ -almost everywhere, and so for μ -almost all $x \in X$,

the operator matrix $G(x) = K(x)^{1/2}F(x)K(x)^{1/2}$ is a positive operator acting on $\bigoplus_1^d \mathcal{H}$, where $K(x)$ is the diagonal operator matrix with diagonal entries $\frac{d\nu}{d\mu}(x)$. Therefore, the (k, ℓ) -entry of $G(x)$ is $g_{k\ell}(x) = \left(\frac{d\nu}{d\mu}(x)\right)^{1/2} f_{k\ell}(x) \left(\frac{d\nu}{d\mu}(x)\right)^{1/2}$ for μ -almost all x . In particular, with $\xi = e_1 \oplus \cdots \oplus e_d \in \bigoplus_1^d \mathcal{H}$, we have

$$0 \leq \langle G(x)\xi, \xi \rangle = \sum_{k,\ell=1}^d \langle g_{k\ell}(x)e_\ell, e_k \rangle$$

for μ -almost all $x \in X$. Hence, assuming $F = [f_{k\ell}]_{k,\ell} \in \mathcal{M}_d(C(X) \otimes \mathcal{B}(\mathcal{H}))$ is positive, we deduce that

$$\begin{aligned} s_{\phi_\nu}([f_{k\ell}]_{k,\ell}) &= \frac{1}{d} \sum_{k,\ell=1}^d \left\langle \left(\int_X f_{k\ell} d\nu \right) e_\ell, e_k \right\rangle \\ &= \frac{1}{d} \int_X \left(\sum_{k,\ell=1}^d \langle g_{k\ell}(x)e_\ell, e_k \rangle \right) d\mu(x) \\ &\geq 0. \end{aligned}$$

That is, ϕ_ν is completely positive. Lastly, because

$$\phi_\nu(1) = \int_X 1 d\nu = \nu(X) = 1,$$

we conclude that ϕ_ν is a unital map. \square

2.4. The Γ -transform. We now formalise the association of $\phi_\nu \in \text{UCP}_{\mathcal{H}}(X)$ with $\nu \in \text{POVM}_{\mathcal{H}}(X)$.

Definition 2.5. Define $\Gamma : \text{POVM}_{\mathcal{H}}(X) \rightarrow \text{UCP}_{\mathcal{H}}(X)$ by

$$\Gamma(\nu) = \phi_\nu.$$

Because the definition of $\phi_\nu(f)$ involves a square root of the principal Radon-Nikodým derivative $\frac{d\nu}{d\mu}$, the transform Γ does not appear to possess any usable affine properties. However, we are able to say how $\Gamma(\nu)$ and $\Gamma(\nu')$ compare if ν' is obtained from ν via composition with an automorphism.

Proposition 2.6. *For every $\nu \in \text{POVM}_{\mathcal{H}}(X)$ and $\alpha \in \text{Aut}(\mathcal{B}(\mathcal{H}))$, the following equation holds:*

$$\Gamma(\alpha \circ \nu) = \alpha \circ \Gamma(\nu) \circ [\text{id}_{C(X)} \otimes \alpha^{-1}].$$

That is, for every $\nu \in \text{POVM}_{\mathcal{H}}(X)$ and unitary $u \in \mathcal{B}(\mathcal{H})$, we have

$$\int_X f d(u^* \nu u) = u^* \left(\int_X u f u^* d\nu \right) u,$$

for every continuous $f : X \rightarrow \mathcal{B}(\mathcal{H})$.

Proof. By Proposition 2.2, $\frac{d(u^*\nu u)}{d\mu} = u^* \frac{d\nu}{d\mu} u$, where $\mu = \frac{1}{d}\text{Tr} \circ \nu = \frac{1}{d}\text{Tr} \circ \alpha \circ \nu$.

Recall that $\int_X f d(u^*\nu u)$ is the unique operator such that, for any state $\rho \in S(\mathcal{H})$,

$$\begin{aligned} \text{Tr} \left(\rho \int_X f d(u^*\nu u) \right) &= \int_X \text{Tr} \left(\rho \left(\frac{d(u^*\nu u)}{d\mu} \right)^{1/2} f \left(\frac{d(u^*\nu u)}{d\mu} \right)^{1/2} \right) d\mu \\ &= \int_X \text{Tr} \left(\rho u^* \left(\frac{d\nu}{d\mu} \right)^{1/2} u f u^* \left(\frac{d\nu}{d\mu} \right)^{1/2} u \right) d\mu \\ &= \int_X \text{Tr} \left(u \rho u^* \left(\frac{d\nu}{d\mu} \right)^{1/2} u f u^* \left(\frac{d\nu}{d\mu} \right)^{1/2} \right) d\mu \\ &= \text{Tr} \left(u \rho u^* \left(\int_X u f u^* d\nu \right) \right) \\ &= \text{Tr} \left(\rho u^* \left(\int_X u f u^* d\nu \right) u \right). \end{aligned}$$

$$\text{Hence, } \int_X f d(u^*\nu u) = u^* \left(\int_X u f u^* d\nu \right) u.$$

□

If one were to seek a similar reformulation of Proposition 2.6 by replacing the automorphism α with a unital, invertible quantum channel \mathcal{E} , then at a purely formal level one would anticipate that

$$(3) \quad \Gamma(\mathcal{E} \circ \nu) = \mathcal{E} \circ \Gamma(\nu) \circ [\text{id}_{C(X)} \otimes \mathcal{E}^{-1}].$$

Note, however, that such a formulation should not require \mathcal{E}^{-1} to be completely positive, for if it were a requirement, then \mathcal{E} would be an automorphism ([10, Corollary 2.3.2], [4, Theorem X.5]), which brings us back to the case of Proposition 2.6. Whether equation (3) holds for arbitrary invertible unital channels \mathcal{E} remains open.

2.5. Non-Principal Radon-Nikodým Derivatives. If $h \in \mathcal{B}(\mathcal{H})_+$, then h^{-1} shall denote the unique positive operator for which $\ker h^{-1} = \ker h$ and $h^{-1}h = hh^{-1} = q$, the projection onto the range of h . Thus, if h is invertible, then h^{-1} is the inverse of h . Once we have this notion for positive operators, a similar notion of generalised inverse for positive operator valued functions can be made.

Theorem 2.7. *The following statements are equivalent for $\nu_1, \nu_2 \in \text{POVM}_\mathcal{H}(X)$:*

- (1) $\nu_2 \ll_{\text{ac}} \nu_1$;
- (2) *there exists a bounded Borel function $g : X \rightarrow \mathcal{B}(\mathcal{H})$, unique up to sets of ν_1 -measure zero, such that*

$$(4) \quad \nu_2(E) = \int_E g d\nu_1, \quad \text{for every } E \in \mathcal{O}(X).$$

If the equivalent conditions above hold and if μ_j is the probability measure induced by ν_j , then $\mu_2 \ll_{\text{ac}} \mu_1$ and

$$(5) \quad g = \left(\frac{d\mu_2}{d\mu_1} \right) \left[\left(\frac{d\nu_1}{d\mu_1} \right)^{-1/2} \left(\frac{d\nu_2}{d\mu_2} \right) \left(\frac{d\nu_1}{d\mu_1} \right)^{-1/2} \right].$$

Proof. Assume that $\nu_2 \ll_{\text{ac}} \nu_1$. If $\mu_1(E) = \frac{1}{d}\text{Tr}(\nu_1(E)) = 0$, then $\nu_1(E) = 0$. By assumption $\nu_2(E) = 0$ and therefore $\mu_2(E) = \frac{1}{d}\text{Tr}(\nu_2(E)) = 0$, which proves that $\mu_2 \ll_{\text{ac}} \mu_1$. Therefore, for any $E \in \mathcal{O}(X)$, we have

$$\nu_2^{(i,j)}(E) = \langle \nu_2(E)e_j, e_i \rangle \ll_{\text{ac}} \langle \nu_1(E)e_j, e_i \rangle = \nu_1^{(i,j)}(E).$$

Coordinate-wise we obtain $\nu_2^{(i,j)} \ll_{\text{ac}} \mu_1$. By applying the chain rule for the classical Radon-Nikodým derivatives we obtain

$$\frac{d\nu_2^{(i,j)}}{d\mu_1} = \frac{d\nu_2^{(i,j)}}{d\mu_2} \frac{d\mu_2}{d\mu_1}$$

Hence,

$$\frac{d\nu_2}{d\mu_1} = \frac{d\nu_2}{d\mu_2} \frac{d\mu_2}{d\mu_1},$$

where $\frac{d\nu_2}{d\mu_2} : X \rightarrow \mathcal{B}(\mathcal{H})_+$ and $\frac{d\mu_2}{d\mu_1} : X \rightarrow \mathbb{R}_+$. With g as above,

$$\left(\frac{d\nu_1}{d\mu_1} \right)^{1/2} g \left(\frac{d\nu_1}{d\mu_1} \right)^{1/2} = \left(\frac{d\mu_2}{d\mu_1} \right) \frac{d\nu_2}{d\mu_2}.$$

Thus, for any state ρ and $E \in \mathcal{O}(X)$,

$$\begin{aligned} \text{Tr} \left(\rho \int_E g d\nu_1 \right) &= \int_E \text{Tr} \left(\rho \left(\frac{d\nu_1}{d\mu_1} \right)^{1/2} g \left(\frac{d\nu_1}{d\mu_1} \right)^{1/2} \right) d\mu_1 \\ &= \int_E \left[\text{Tr} \left(\rho \frac{d\nu_2}{d\mu_2} \right) \right] \frac{d\mu_2}{d\mu_1} d\mu_1 \\ &= \int_E \text{Tr} \left(\rho \frac{d\nu_2}{d\mu_2} \right) d\mu_2 \\ &= \text{Tr}(\rho \nu_2(E)). \end{aligned}$$

Therefore, by definition of the integral, $\nu_2(E) = \int_E g d\nu_1$ for every $E \in \mathcal{O}(X)$.

Passing to the d^2 coordinate measures and using the uniqueness of the classical Radon-Nikodým derivative, one deduces that g is unique up to sets of ν_1 -measure zero.

Conversely, assume such a function $g : X \rightarrow \mathcal{B}(\mathcal{H})$ exists such that

$$\nu_2(E) = \int_E g d\nu_1, \quad \text{for every } E \in \mathcal{O}(X).$$

If $\nu_1(E) = 0$, then $\nu_2(E) = \int_E g d\nu_1 = 0$ and thus $\nu_2 \ll_{\text{ac}} \nu_1$. \square

The function g in (5) is called a *non-principal Radon-Nikodým derivative* of ν_2 with respect to ν_1 .

3. TOPOLOGY OF $\text{POVM}_{\mathcal{H}}(X)$

In classical probability, $P(X)$ is a weak*-closed subset of the unit sphere of the dual space of the abelian C^* -algebra $C(X)$. Hence, by the Banach-Alaoglu Theorem, $P(X)$ is compact. In this section we introduce an analogous topology on $\text{POVM}_{\mathcal{H}}(X)$ so that it not only is a compact topological space, but also has the property that the POVMs with finite support are dense in $\text{POVM}_{\mathcal{H}}(X)$.

Definition 3.1. Define $\Omega_0 : \mathbf{E}_{\mathcal{H}}(X) \rightarrow \text{POVM}_{\mathcal{H}}(X)$ by

$$\Omega_0 \left(\sum_{j=1}^n t_j^* \varrho_{x_j} t_j \right) = \sum_{j=1}^n \delta_{x_j} t_j^* t_j.$$

Proposition 3.1. *The map Ω_0 is C^* -affine.*

Proof. If $\phi = \sum_{j=1}^n t_j^* \varrho_{x_j} t_j$, then $\Omega_0(\phi) = \sum_{j=1}^n \delta_{x_j} t_j^* t_j$. Let $a \in \mathcal{B}(\mathcal{H})$ and note that

$$\Omega_0(a^* \phi a) = \sum_{j=1}^n \delta_{x_j} a^* t_j^* t_j a = a^* \Omega_0(\phi) a.$$

Thus, if $\phi_1, \dots, \phi_m \in \mathbf{E}_{\mathcal{H}}$ and if $a_1, \dots, a_m \in \mathcal{B}(\mathcal{H})$ are C^* -convex coefficients, then

$$\Omega_0 \left(\sum_{k=1}^m a_k^* \phi_k a_k \right) = \sum_{k=1}^m a_k^* \Omega_0(\phi_k) a_k \in \text{POVM}_{\mathcal{H}}(X),$$

which completes the proof. \square

We aim to show that Ω_0 admits a C^* -affine extension to $\text{UCP}_{\mathcal{H}}(X)$. Before discussing the extension map, we will describe the topologies of $\text{UCP}_{\mathcal{H}}(X)$ and $\text{POVM}_{\mathcal{H}}(X)$. Because topologies can be characterised by how nets converge, we begin with the BW-topology of $\text{UCP}_{\mathcal{H}}(X)$ and use Γ to induce a topology on $\text{POVM}_{\mathcal{H}}(X)$.

Definition 3.2. (The bounded-weak topology) A net $\{\phi_\gamma\}_{\gamma \in \Lambda} \subset \text{UCP}_{\mathcal{H}}(X)$ converges to $\phi \in \text{UCP}_{\mathcal{H}}(X)$ if $\lim_{\gamma} \|\phi_\gamma(f) - \phi(f)\| = 0$ for all $f \in C(X) \otimes \mathcal{B}(\mathcal{H})$.

In the BW-topology, $\text{UCP}_{\mathcal{H}}(X)$ is a compact space [2, 23]. We use the topology of $\text{UCP}_{\mathcal{H}}(X)$ to induce a topology on $\text{POVM}_{\mathcal{H}}(X)$ as follows.

Definition 3.3. A net $\{\nu_\gamma\}_{\gamma \in \Lambda} \subset \text{POVM}_{\mathcal{H}}(X)$ converges to $\nu \in \text{POVM}_{\mathcal{H}}(X)$ if

$$\int_X f d\nu = \lim_{\gamma} \int_X f d\nu_\gamma \quad \text{for every } f \in C(X) \otimes \mathcal{B}(\mathcal{H}).$$

That is, $\nu_\gamma \rightarrow \nu$ if the net $\{\Gamma(\nu_\gamma)\}_{\gamma \in \Lambda}$ converges to $\Gamma(\nu)$ in $\text{UCP}_{\mathcal{H}}$.

Theorem 3.2. *There exists a C^* -affine function $\Omega : \text{UCP}_{\mathcal{H}}(X) \rightarrow \text{POVM}_{\mathcal{H}}(X)$ such that*

- (1) $\Omega|_{\mathbf{E}_{\mathcal{H}}(X)} = \Omega_0$ and
- (2) $\Omega \circ \Gamma = \text{id}_{\text{POVM}_{\mathcal{H}}(X)}$.

Proof. We first prove that $E_{\mathcal{H}}(X)$ is BW-dense in $UCP_{\mathcal{H}}(X)$. Because $UCP_{\mathcal{H}}$ is BW-compact and convex, the Krein-Milman theorem asserts that the convex hull of the extreme points of $UCP_{\mathcal{H}}(X)$ is a dense subset of $UCP_{\mathcal{H}}(X)$. If $\phi \in \text{ext}(UCP_{\mathcal{H}}(X))$, then there exist irreducible representations $\pi_j : C(X) \otimes \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}_{\pi_j})$ and an isometry $v : \mathcal{H} \rightarrow \mathcal{H}_{\pi} = \bigoplus_{j=1}^m \mathcal{H}_{\pi_j}$ such that $\phi = v^* \pi v$ and the linear map $y \mapsto v^* y v$ is one-to-one on the commutant \mathcal{N} of $\pi(C(X) \otimes \mathcal{B}(\mathcal{H}))$ [2, Theorem 1.4.6]. Because $\dim \mathcal{H} = d < \infty$, every irreducible representation π_j of $C(X) \otimes \mathcal{B}(\mathcal{H})$ takes place on a Hilbert space \mathcal{H}_{π_j} of dimension d and has the form $\pi = u_j^* \varrho_{x_j} u_j$, where $\varrho_{x_j} \in UCP_{\mathcal{H}}(X)$ is spectral and $u_j : \mathcal{H}_{\pi_j} \rightarrow \mathcal{H}$ is unitary. Because \mathcal{H}_{π_j} and \mathcal{H} are of dimension d , we may assume without loss of generality that $\mathcal{H}_{\pi_j} = \mathcal{H}$ for every j , whence $\pi_j = \alpha_j \circ \varrho_{x_j}$ for the automorphism $\alpha_j \in \text{Aut}(\mathcal{B}(\mathcal{H}))$ given by $\alpha_j(z) = u_j^* z u_j$. If $q_j \in \mathcal{B}(\mathcal{H}_{\pi})$ is the projection of \mathcal{H}_{π} onto the direct summand $\mathcal{H}_{\pi_j} = \mathcal{H}$, then with $a_i = u_i q_i v \in \mathcal{B}(\mathcal{H})$ we obtain

$$\phi = \sum_{j=1}^m a_j^* \varrho_{x_j} a_j,$$

which is an element of $E_{\mathcal{H}}(X)$. Therefore, $E_{\mathcal{H}}(X)$ contains the extreme points of $UCP_{\mathcal{H}}(X)$. By the convexity of $E_{\mathcal{H}}(X)$ and the Krein-Milman theorem we deduce that $E_{\mathcal{H}}(X)$ is BW-dense in $UCP_{\mathcal{H}}(X)$.

Now assume that $\phi \in UCP_{\mathcal{H}}(X) \setminus E_{\mathcal{H}}(X)$. Thus, there exists a net $\{\phi_{\gamma}\}_{\gamma \in \Lambda} \subset E_{\mathcal{H}}$ such that $\|\phi_{\gamma}(f) - \phi(f)\| \rightarrow 0$ for all $f \in C(X) \otimes \mathcal{B}(\mathcal{H})$. For each pair (i, j) let $(\phi_{\gamma})_{ij} \in C(X)^*$ be given by

$$(\phi_{\gamma})_{ij}(g) = \langle \phi_{\gamma}(g \otimes e_{ij}) e_j, e_i \rangle$$

for all $g \in C(X)$. Thus,

$$\phi_{\gamma} = \sum_{i,j=1}^d (\phi_{\gamma})_{ij} \otimes e_{ij}.$$

By the classical Riesz Representation Theorem, there exists a complex measure $\nu_{\gamma_{ij}}$ on the Borel sets of X such that

$$(\phi_{\gamma})_{ij}(g) = \int_X g d\nu_{\gamma_{ij}} \text{ for all } g \in C(X).$$

Since $\phi_{\gamma}(f) \rightarrow \phi(f)$ for all $f \in C(X) \otimes \mathcal{B}(\mathcal{H})$ we have that $(\phi_{\gamma})_{ij}(g) \rightarrow \phi_{ij}(g)$ for all $g \in C(X)$ and pairs (i, j) . Therefore $(\nu_{\gamma})_{ij} \rightarrow \nu_{ij}$. Define

$$\nu = \sum_{i,j=1}^d \nu_{ij} \otimes e_{ij} \in \text{POVM}_{\mathcal{H}}(X)$$

and let $\Omega(\phi) = \nu$. This is well defined as ν is independent of the choice of approximating net $\{\phi_{\gamma}\}$.

Let $\phi_1, \dots, \phi_m \in UCP_{\mathcal{H}}(X)$ and suppose that $\{\phi_{\gamma_j}\}_{\gamma_j \in \Lambda_j} \subset E_{\mathcal{H}}(X)$ is an approximating net for ϕ_j , for each j . Let $\Lambda = \Lambda_1 \times \dots \times \Lambda_m$, ordered as a directed set in the natural way. If $t_1, \dots, t_m \in \mathcal{B}(\mathcal{H})$ are C^* -convex coefficients, then with respect to the natural induced directed-set ordering on $\Lambda_1 \times \dots \times \Lambda_m$, the ucp maps

$$\left\{ \sum_{j=1}^m t_j^* \phi_{\gamma_j} t_j \right\}_{\gamma=(\gamma_1, \dots, \gamma_m) \in \Lambda}$$

form an approximating net for $\sum_{j=1}^m t_j^* \phi_j t_j$. Because Ω_0 is C^* -affine,

$$\begin{aligned}\Omega\left(\sum_{j=1}^m t_j^* \phi_j t_j\right) &= \lim_{\gamma} \Omega_0\left(\sum_{j=1}^m t_j^* \phi_{\gamma_j} t_j\right) \\ &= \lim_{\gamma} \sum_{j=1}^m t_j^* \Omega_0(\phi_{\gamma_j}) t_j \\ &= \sum_{j=1}^m t_j^* \Omega(\phi_j) t_j,\end{aligned}$$

which establishes the extension Ω of Ω_0 .

Observe that if $\nu = \sum_{j=1}^n \delta_{x_j} h_j$, then $\Omega \circ \Gamma(\nu) = \nu$. Suppose now that $\nu \in \text{POVM}_{\mathcal{H}}(X)$ is arbitrary and let $\{\phi_{\gamma}\}_{\gamma}$ be a net in $E_{\mathcal{H}}(X)$ BW-convergent to $\Gamma(\nu)$. For each γ , let $\nu_{\gamma} = \Omega \phi_{\gamma}$ so that $\Omega \Gamma(\nu_{\gamma}) = \nu_{\gamma}$. As shown earlier, another realisation of ν is via

$$\nu = \sum_{i,j=1}^d \nu_{ij} \otimes e_{ij},$$

where ν_{ij} is attained in the dual space of $C(X)$ as the limit of the net of measures induced by the linear functionals $(\phi_{\gamma})_{ij}(g) = \langle \phi(g \otimes e_{ij}) e_j, e_i \rangle$. Thus $\int_X f d\nu_{\gamma} \rightarrow \int_X f d\nu$ for all $f \in C(X) \otimes \mathcal{B}(\mathcal{H})$ and $\Omega \Gamma(\nu) = \nu$. \square

Corollary 3.3. $\text{POVM}_{\mathcal{H}}(X)$ is a compact space.

Proof. If $\{\nu\}_{\gamma}$ is a net in $\text{POVM}_{\mathcal{H}}(X)$, then $\{\Gamma \nu_{\gamma}\}_{\gamma}$ is a net in $\text{UCP}_{\mathcal{H}}(X)$. Because $\text{UCP}_{\mathcal{H}}(X)$ is compact, there is a subnet $\{\Gamma \nu_{\gamma_j}\}_j$ which is convergent, say to ϕ . Thus, if $\nu = \Omega \phi$, then $\{\nu_{\gamma_j}\}_j$ converges to ν . Hence, every net in $\text{POVM}_{\mathcal{H}}(X)$ admits a convergent subnet, and so $\text{POVM}_{\mathcal{H}}(X)$ is compact. \square

A consequence of the Krein-Milman Theorem and the proof of Theorem 3.2 is:

Corollary 3.4. The set of all quantum measurements with finite support is dense in $\text{POVM}_{\mathcal{H}}(X)$.

Operators z acting on finite-dimensional Hilbert spaces \mathcal{K} admit polar decompositions of the form $z = u|z|$, where $|z| = (z^* z)^{1/2}$ and $u \in \mathcal{B}(\mathcal{K})$ is unitary. (The unitary need not be unique if z is not invertible.) Therefore,

$$\begin{aligned}\Gamma \circ \Omega_0\left(\sum_{j=1}^n t_j^* \varrho_{x_j} t_j\right) &= \Gamma\left(\sum_{j=1}^n \delta_{x_j} t_j^* t_j\right) \\ &= \sum_{j=1}^n (t_j^* t_j)^{1/2} \varrho_{x_j} (t_j^* t_j)^{1/2} \\ &= \sum_{j=1}^n |t_j| (\alpha_j \circ \varrho_{x_j}) |t_j|,\end{aligned}$$

where $t_j = u_j |t_j|$ is a polar decomposition of t_j and $\alpha_j \in \text{Aut}(\mathcal{B}(\mathcal{H}))$ is given by $\alpha_j(z) = \underline{u_j^* z u_j}$. Thus, if $\phi \in E_{\mathcal{H}}^+(X)$, we have $\Gamma \circ \Omega_0(\phi) = \phi$.

Let $E_{\mathcal{H}}^+(X)$ denote the BW-closure of $E_{\mathcal{H}}^+(X)$ in $\text{UCP}_{\mathcal{H}}(X)$.

Corollary 3.5. (Riesz Representation Theorem) *For every $\phi \in \overline{E_{\mathcal{H}}^+(X)}$ there is a unique $\nu \in \text{POVM}_{\mathcal{H}}(X)$ such that*

$$\phi(f) = \int_X f d\nu \quad \text{for every } f \in C(X) \otimes \mathcal{B}(\mathcal{H}).$$

Proof. Let $\nu = \Omega\phi$ so that $\Gamma\nu = \Gamma\Omega\phi = \phi$. If ν' is another POVM for which $\Gamma(\nu') = \phi$, then $\nu' = \Omega\Gamma\nu' = \Omega\phi = \nu$. \square

4. CLASSICAL RANDOMNESS

In this section we establish the following result.

Theorem 4.1. *The following statements are equivalent for $\nu \in \text{POVM}_{\mathcal{H}}(X)$:*

- (1) ν is an extreme point of $\text{POVM}_{\mathcal{H}}(X)$;
- (2) there exist distinct $x_1, \dots, x_m \in X$ and $h_1, \dots, h_m \in \mathcal{B}(\mathcal{H})_+$ such that
 - (a) the subspaces $\text{ran } h_1, \dots, \text{ran } h_m$ are weakly independent and
 - (b) $\nu = \sum_{j=1}^m \delta_{x_j} h_j$.

In the case of finite X , Theorem 4.1 is already known [9, 22]. Our contribution is to show that the case of arbitrary X reduces to the case of finite X (Lemmas 4.2 and 4.3). However, for completeness, we include a full proof of Theorem 4.1, adapting the elegant arguments of D’ariano, Lo Presti, and Perinotti [9]. The concept of weak independence is defined formally below.

Definition 4.1. Subspaces $\mathcal{L}_1, \dots, \mathcal{L}_n \subset \mathcal{H}$ are weakly independent if, for any $t_1, \dots, t_n \in \mathcal{B}(\mathcal{H})$ such that (i) $\text{ran } t_j + \text{ran } t_j^* \subset \mathcal{L}_j$ for each j , and (ii) $t_1 + \dots + t_n = 0$, then necessarily each $t_j = 0$.

The remainder of this section is devoted to the proof of Theorem 4.1 and a discussion of some of its consequences.

Recall that if $K \subset X$ is a closed subset, then the σ -algebra $\mathcal{O}(K)$ of Borel sets of K is given by $\mathcal{O}(K) = \{K \cap E : E \in \mathcal{O}(X)\}$.

Lemma 4.2. *Assume $K_\nu \subset X$ is the support of $\nu \in \text{POVM}_{\mathcal{H}}(X)$. Then $\nu \in \text{ext}(\text{POVM}_{\mathcal{H}}(X))$ if and only if the restriction $\nu|_{\mathcal{O}(K_\nu)}$ of ν to $\mathcal{O}(K_\nu)$ is an extreme point of $\text{POVM}_{\mathcal{H}}(K_\nu)$.*

Proof. Assume that $\nu \in \text{ext}(\text{POVM}_{\mathcal{H}}(X))$. Let $\nu_0, \nu_1, \nu_2 \in \text{POVM}_{\mathcal{H}}(K_\nu)$ and such that $\nu|_{\mathcal{O}(K_\nu)} = \nu_0 = \frac{1}{2}(\nu_1 + \nu_2)$. Define $\tilde{\nu}_j : \mathcal{O}(X) \rightarrow \mathcal{B}(\mathcal{H})$ by $\tilde{\nu}_j(E) = \nu_j(E \cap K_\nu)$ for all $E \in \mathcal{O}(X)$, to obtain $\tilde{\nu}_j \in \text{POVM}_{\mathcal{H}}(X)$. Because K_ν is the support of ν , $\nu(E) = \nu(E \cap K_\nu)$ for all $E \in \mathcal{O}(X)$; thus, $\nu = \frac{1}{2}(\tilde{\nu}_1 + \tilde{\nu}_2)$, and so $\nu = \tilde{\nu}_1 = \tilde{\nu}_2$, and so $\nu_0 = \nu_1 = \nu_2$.

Conversely, assume that $\nu_0 = \nu|_{\mathcal{O}(K_\nu)}$ is an extreme point of $\text{POVM}_{\mathcal{H}}(K_\nu)$. Let $\nu = \frac{1}{2}(\nu_1 + \nu_2)$ for $\nu_1, \nu_2 \in \text{POVM}_{\mathcal{H}}(X)$. If $E \in \mathcal{O}(X)$ satisfies $\nu(E) = 0$ then $0 = \nu(E) \geq \frac{1}{2}\nu_j \geq 0$ implies that $\nu_j = 0$. Thus, $\nu_j \ll_{\text{ac}} \nu$. If we show that

the support of each ν_j is contained in the support of ν , then we conclude that $\nu_1 = \nu_2 = \nu$.

Thus, it remains to prove that if $\omega, \nu \in \text{POVM}_{\mathcal{H}}(X)$ is such that $\omega \ll_{\text{ac}} \nu$, then $K_\omega \subset K_\nu$. To this end, let $U = (X \setminus K_\nu) \cap (X \setminus K_\omega)$, which is open, and let $K = (X \setminus U) \cap K_\omega$, which is closed. Thus,

$$\omega(X \setminus K) = \omega(U \cup (X \setminus K_\omega)) \leq \omega(U) + \omega(X \setminus K_\omega) = \omega(U).$$

Now since $U \subset X \setminus K_\nu$, we have $\nu(U) \leq \nu(X \setminus K_\nu) = 0$. Thus, $\omega \ll_{\text{ac}} \nu$ implies that $\omega(U) = 0$ and so $\omega(X \setminus K) = 0$. Hence, $K \subset K_\omega$ and $\omega(X \setminus K) = 0$ which implies that $K = K_\omega$ by definition of support and by the above arguments. Hence, $K_\omega = K = (X \setminus U) \cap K_\omega = (K_\nu \cup K_\omega) \cap K_\omega$ implies that $K_\omega \subset K_\nu$. \square

Lemma 4.3. *If $\nu \in \text{POVM}_{\mathcal{H}}(X)$ is an extreme point, then the support of ν is a finite set.*

Proof. Assume, contrary to what we aim to prove, that the support of ν is an infinite set. By Lemma 4.2, we may replace X with the support of ν , and so we assume without loss of generality that $X = K_\nu$. The argument below is inspired by the proof of the main result of [12].

Let $\mu = \frac{1}{d} \text{Tr} \circ \nu$ and consider $L^1(X, \mu)$. Because μ and ν are mutually absolutely continuous, they have the same support X . Thus $L^1(X, \mu)$ is an infinite-dimensional Banach space.

Let $\kappa_{ij}(x) = \left\langle \left(\frac{d\nu}{d\mu} \right) e_j, e_i \right\rangle$, for $1 \leq i, j \leq d$, and let

$$\begin{aligned} Q_0 &= \left\{ \text{Tr} \left[\left(\frac{d\nu}{d\mu} \right)^{1/2} \rho \left(\frac{d\nu}{d\mu} \right)^{1/2} \right] : \rho \in S(\mathcal{H}) \right\} \\ &= \left\{ \text{Tr} \left(\rho \frac{d\nu}{d\mu} \right) : \rho \in S(\mathcal{H}) \right\} \\ &\subset \text{Span} \{ \kappa_{ij} : 1 \leq i, j \leq d \}. \end{aligned}$$

Let $Q = \text{Span } Q_0$; thus, Q is a subspace of $L^1(X, \mu)$ of dimension at most d^2 . Because $L^1(X, \mu)$ has infinite dimension, the annihilator of Q in the dual space $L^1(X, \mu)^* = L^\infty(X, \mu)$ has infinite dimension. Hence, there exists $\varphi \in L^\infty(X, \mu)$ such that $\|\varphi\| = 1$ and $\int_X \varphi \psi d\mu = 0$ for all $\psi \in Q$. Let $\Phi = \sum_{j=1}^d \varphi \otimes e_{jj}$; thus, for any state $\rho \in S(\mathcal{H})$,

$$\int_X \text{Tr} \left[\left(\frac{d\nu}{d\mu} \right)^{1/2} \rho \left(\frac{d\nu}{d\mu} \right)^{1/2} \Phi \right] d\mu = \int_X \text{Tr} \left(\varphi \rho \frac{d\nu}{d\mu} \right) d\mu = 0.$$

Hence,

$$(6) \quad \int_X \Phi d\nu = 0.$$

Define $\tilde{\nu} : \mathcal{O}(X) \rightarrow \mathcal{B}(\mathcal{H})$ by $\tilde{\nu}(E) = \int_E \Phi d\nu$. Let $\nu_1 = \nu + \tilde{\nu}$ and $\nu_2 = \nu - \tilde{\nu}$. Note that

$$\nu_1(E) = \int_E d(\nu + \tilde{\nu}) = \int_E d\nu + \int_E \Phi d\nu = \int_E (1 + \Phi) d\nu.$$

Since $1 + \Phi$ is positive for all $x \in X$, this final integral above is a positive operator. Likewise $\nu_2(E)$ is positive. Further, by equation (6),

$$\nu_1(X) = \int_X d(\nu + \tilde{\nu}) = \nu(X) + \int_X \Phi d\nu = \nu(X) + 0 = 1.$$

Hence, $\nu_1, \nu_2 \in \text{POVM}_{\mathcal{H}}(X)$ and $\nu = \frac{1}{2}\nu_1 + \frac{1}{2}\nu_2$. Since $\nu_1 \neq \nu$ (because $\varphi \neq 0$), we have $\nu \notin \text{ext}(\text{POVM}_{\mathcal{H}}(X))$. \square

We are now prepared for the proof of Theorem 4.1, using an adaptation of the arguments of the D'ariano, Lo Presti, and Perinotti [9]. Recall that we aim to prove that the following statements are equivalent for $\nu \in \text{POVM}_{\mathcal{H}}(X)$:

- (1) ν is an extreme point of $\text{POVM}_{\mathcal{H}}(X)$;
- (2) there exist distinct $x_1, \dots, x_n \in X$ and $h_1, \dots, h_n \in \mathcal{B}(\mathcal{H})_+$ such that
 - (a) the subspaces $\text{ran } h_1, \dots, \text{ran } h_n$ are weakly independent and
 - (b) $\nu = \sum_{j=1}^n \delta_{x_j} h_j$.

Proof. Assume (1). Thus, $\nu \in \text{ext}(\text{POVM}_{\mathcal{H}}(X))$. By Lemma 4.2, we may replace X with the support K_ν of ν , which by Lemma 4.3 is a finite set if $\nu \in \text{ext}(\text{POVM}_{\mathcal{H}}(X))$. Thus, without loss generality we may assume that X is a finite

set, say $X = \{x_1, \dots, x_n\}$, and that the support of ν is X . Hence, $\nu = \sum_{j=1}^n \delta_{x_j} h_j$.

Suppose that $t_1, \dots, t_n \in \mathcal{B}(\mathcal{H})$ satisfy $\sum_{j=1}^m t_j = 0$ and $\text{ran } t_j + \text{ran } t_j^* \subset \text{ran } h_j$ for all

$1 \leq j \leq n$. Also assume, contrary to what we aim to prove, that not every t_j is zero. If every t_j is hermitian, then let $g_j = t_j$ for all j and $g_k \neq 0$ for some k ; if not all t_1, \dots, t_n are hermitian, then there exists k with $\Im(t_k) \neq 0$ and in this situation we take $g_j = \Im(t_j)$ for all $1 \leq j \leq n$. (The imaginary part of $s \in \mathcal{B}(\mathcal{H})$ is the hermitian operator $\Im(s) = \frac{1}{2i}(s - s^*)$.) With this choice of operators g_j , we have

$\text{rang}_j \subset \text{ran } h_j$ for all $1 \leq j \leq n$, $g_k \neq 0$ for some k , and $\sum_{j=1}^n g_j = 0$. For each j write $\mathcal{H} = \ker h_j \oplus \text{ran } h_j$ so that

$$h_j = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{h}_j \end{bmatrix} \quad \text{and} \quad g_j = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{g}_j \end{bmatrix},$$

where $\tilde{h}_j \in \mathcal{B}(\text{ran } h_j)$ is positive and invertible. (If h_j itself is invertible, then we do not use a 2×2 operator matrix and simply take $\tilde{h}_j = h_j$.) Let

$$\mathcal{Z}_\varepsilon = \left\{ 1 \pm \epsilon \lambda \mid \lambda \in \bigcup_{j=1}^n \sigma(\tilde{h}_j^{-1/2} \tilde{g}_j \tilde{h}_j^{-1/2}) \right\},$$

where $\sigma(z)$ denotes the spectrum of an operator z . Because \mathcal{Z}_ε is a finite set, there exists $\varepsilon > 0$ so that $\mathcal{Z}_\varepsilon \subset (0, \infty)$. Hence,

$$h_j \pm \varepsilon g_j = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{h}_j^{1/2} (1 \pm \varepsilon \tilde{h}_j^{-1/2} \tilde{g}_j \tilde{h}_j^{-1/2}) \tilde{h}_j^{1/2} \end{bmatrix},$$

which is positive by the choice of $\varepsilon > 0$. Now let $\gamma = \sum_{j=1}^n \delta_{x_j} g_j$ to obtain $\nu \pm \varepsilon\gamma \in \text{POVM}_{\mathcal{H}}(X)$. If $\nu_1 = \nu + \varepsilon\gamma$ and $\nu_2 = \nu - \varepsilon\gamma$, then ν_1, ν_2 are distinct elements of $\text{POVM}_{\mathcal{H}}(X)$ and ν is their midpoint, in contradiction to the hypothesis $\nu \in \text{ext}(\text{POVM}_{\mathcal{H}}(X))$. Hence, it must be that all of the operators t_j are zero, which is to say the ranges of h_1, \dots, h_n are weakly independent.

Conversely, assume (2). Thus, $\nu = \sum_{j=1}^n \delta_{x_j} h_j$ for distinct $x_1, \dots, x_n \in X$ and positive operators $h_1, \dots, h_n \in \mathcal{B}(\mathcal{H})$ with weakly independent ranges. The support of ν is given by $K_\nu = \{x_1, \dots, x_n\}$. By Lemma 4.2, ν is an extreme point of $\text{POVM}_{\mathcal{H}}(X)$ if and only if ν is an extreme point of $\text{POVM}_{\mathcal{H}}(K_\nu)$. Hence, we assume without loss of generality that $X = K_\nu$. If, contrary to what we wish to prove, $\nu \notin \text{ext}(\text{POVM}_{\mathcal{H}}(X))$, then in the real vector space W of countably additive functions $v : \mathcal{O}(X) \rightarrow \mathcal{B}(\mathcal{H})_{\text{sa}}$ there exists $\varepsilon > 0$ and $\gamma \in W$ such that $\omega^\pm = \nu \pm \varepsilon\gamma \in \text{POVM}_{\mathcal{H}}(X)$. Therefore, there exist $g_1, \dots, g_n \in \mathcal{B}(\mathcal{H})_{\text{sa}}$ such that $\gamma = \sum_{j=1}^n \delta_{x_j} g_j$ and $g_k \neq 0$ for at least one k . Because

$$1 = \omega^\pm(X) = \nu(X) \pm \varepsilon\gamma(X) = 1 \pm \varepsilon \sum_{j=1}^n g_j,$$

we obtain $\sum_{j=1}^n g_j = 0$. And because $\omega^\pm(E) \in \text{Eff}(\mathcal{H})$ we obtain through evaluation at x_j that $h_j \pm \varepsilon g_j \in \mathcal{B}(\mathcal{H})_+$. Now write $\mathcal{H} = \ker h_j \oplus \text{ran } h_j$ so that

$$h_j = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{h}_j \end{bmatrix} \quad \text{and} \quad g_j = \begin{bmatrix} a_j & y_j \\ y_j^* & \tilde{g}_j \end{bmatrix},$$

where $a_j \in \mathcal{B}(\ker h_j)$ is hermitian and $\tilde{h}_j \in \mathcal{B}(\text{ran } h_j)$ is positive and invertible. Because $h_j + \varepsilon g_j$ is positive, a_j is necessarily positive. But $h_j - \varepsilon g_j$ positive implies that $-a_j$ is positive. Hence $a_j = 0$. The positivity of $h_j + \varepsilon g_j = \begin{bmatrix} 0 & \varepsilon y_j \\ \varepsilon y_j^* & \varepsilon \tilde{g}_j \end{bmatrix}$ yields $y_j = 0$, and so $g_j \xi = 0$ for every vector $\xi \in \mathcal{H}$ for which $h_j \xi = 0$. That is, $\ker h_j \subset \ker g_j$ and so $\text{ran } g_j \subset \text{ran } h_j$. We conclude that $\text{ran } h_1, \dots, \text{ran } h_n$ are not weakly independent, contrary to hypothesis. Hence, it must be that no such function γ exists, which is to say that ν is an extreme point of $\text{POVM}_{\mathcal{H}}(X)$. \square

Corollary 4.4. *If ν is an extreme point of $\text{POVM}_{\mathcal{H}}(X)$, then so is $\alpha \circ \nu$ for every automorphism α of $\mathcal{B}(\mathcal{H})$.*

Proof. One need only note that if $u \in \mathcal{B}(\mathcal{H})$ is unitary and $h_1, \dots, h_n \in \mathcal{B}(\mathcal{H})$ are positive, then h_1, \dots, h_n have weakly independent ranges if and only if $u^* h_1 u, \dots, u^* h_n u$ have weakly independent ranges. \square

Although the function Γ does not exhibit affine properties, it does map extremal elements to extremal elements.

Corollary 4.5. *If ν is an extreme point of $\text{POVM}_{\mathcal{H}}(X)$, then $\Gamma(\nu)$ is an extreme point of $\text{UCP}_{\mathcal{H}}(X)$.*

Proof. Assume that $\nu \in \text{ext}(\text{POVM}_{\mathcal{H}}(X))$ and that $\phi_{\nu} = \Gamma(\nu)$. Hence, by Theorem 4.1, $\phi_{\nu} = \sum_{j=1}^n h_j^{1/2} \varrho_{x_j} h_j^{1/2}$ for some distinct $x_1, \dots, x_n \in X$ and positive operators h_1, \dots, h_n with weakly independent ranges. Each ϱ_{x_j} is an irreducible representation of $C(X) \otimes \mathcal{B}(\mathcal{H})$ on \mathcal{H} and, because the points $x_1, \dots, x_n \in X$ are distinct, no two irreducible representations ϱ_{x_i} and ϱ_{x_j} corresponding to distinct points x_i and x_j are unitarily equivalent. By weak independence, every $h_j \neq 0$; and by irreducibility each completely positive map $\phi_j : C(X) \otimes \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ of the form

$$\phi_j = h_j^{1/2} \varrho_{x_j} h_j^{1/2}$$

generates an extremal ray of the cone of all completely positive linear maps $C(X) \otimes \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ [2, Corollary 1.4.3]. Therefore, by [2, Lemma 1.4.9], $\phi_{\nu} = \sum_{j=1}^n \phi_j$ is an extremal ucp map. \square

5. NONCLASSICAL RANDOMNESS

A convex combination $\sum_{j=1}^m \lambda_j \nu_j$ of $\nu_1, \dots, \nu_m \in \text{POVM}_{\mathcal{H}}(X)$ is a random POVM if one views the set $\{\lambda_1, \dots, \lambda_m\}$ of convex coefficients as a probability distribution. The notion of classical randomness enters quantum measurement through probabilistic (that is, convex combinations) mixtures of other quantum measurements.

In nonclassical convexity, there are corresponding notions of randomness afforded by C^* -convex coefficients. We mention below two sources of randomness.

First, assume that $a = (a_1, \dots, a_m)$ is a tuple of C^* -convex coefficients $a_j \in \mathcal{B}(\mathcal{H})$. Because \mathcal{H} has finite dimension, the subspaces $\ker a_j$ and $\ker a_j^*$ have equal dimension for each j . Hence, the isometry $v : \mathcal{H} \rightarrow \bigoplus_{j=1}^m \mathcal{H}$ defined by

$$v\xi = \bigoplus a_j \xi, \quad \xi \in \mathcal{H},$$

extends to a unitary $u \in \mathcal{B}\left(\bigoplus_{j=1}^m \mathcal{H}\right)$ [1, Corollary 2.2]. Conversely, expressing any unitary u acting on $\bigoplus_{j=1}^m \mathcal{H}$ as an $m \times m$ matrix of operators and by selecting the operators that appear in any single column of u , one obtains a tuple of C^* -convex coefficients. Hence, by endowing the unitary group $\mathcal{U}\left(\bigoplus_{j=1}^m \mathcal{H}\right)$ with Haar measure, every m -tuple of C^* -convex coefficients is determined by a random unitary.

Second, if $a = (a_1, \dots, a_m)$ is a tuple of C^* -convex coefficients, then a induces a quantum channel $\mathcal{E}_a : S(\mathcal{H}) \rightarrow S(\mathcal{H})$ via

$$\mathcal{E}_a(\rho) = \sum_{j=1}^m a_j \rho a_j^*, \quad \rho \in S(\mathcal{H}).$$

Conversely, every quantum channel induces a tuple of C^* -convex coefficients. Hence, any source of randomness for channels (and there are many to choose from—see, for example, [3, §14.7]), is a source of randomness for nonclassical convex combinations.

We earlier introduced the notion of a proper C^* -convex hull $C_p^*\text{conv } R$ of a set R by using C^* -convex combinations of elements of R using only invertible C^* -convex coefficients.

Definition 5.1. A *coarsening* of quantum measurements $\nu_1, \dots, \nu_n \in \text{POVM}_{\mathcal{H}}(X)$ is any measurement $\nu \in \text{POVM}_{\mathcal{H}}(X)$ that satisfies

$$\nu \in C_p^*\text{conv}(\{\nu_1, \dots, \nu_n\}).$$

The use of invertible C^* -convex combinations in the definition of coarsening of measurements ensures that a certain level of information is conserved through the process of coarsening.

We use the notation

$$\nu \ll_{\text{crse}} (\nu_1, \dots, \nu_m)$$

to indicate that ν is a coarsening of ν_1, \dots, ν_m .

Definition 5.2. A quantum measurement ν is *fine* if $\nu \ll_{\text{crse}} (\nu_1, \dots, \nu_m)$ occurs only if each measurement ν_j is unitarily equivalent to ν .

Although the definition of fine measurement above appears to differ from the notion of sharp measurement defined earlier (Definition 1.1), the following theorem shows that the two concepts in fact are the same.

Theorem 5.1. *The following statements are equivalent for $\nu \in \text{POVM}_{\mathcal{H}}(X)$:*

- (1) ν is a C^* -extreme point of $\text{POVM}_{\mathcal{H}}(X)$;
- (2) ν is fine;
- (3) ν is sharp—that is, there exist distinct $x_1, \dots, x_n \in X$ and pairwise-orthogonal projections $q_1, \dots, q_n \in \mathcal{B}(\mathcal{H})$ such that

$$\nu = \sum_{j=1}^n \delta_{x_j} q_j.$$

The equivalence of (1) and (2) in Theorem 5.1 is trivial, as the definition of fineness herein is precisely the definition of C^* -extreme point. Some preparatory results are required to show the third equivalence.

Via the identification of $C(X)$ as a unital C^* -subalgebra of $C(X) \otimes \mathcal{B}(\mathcal{H})$, each $\phi \in \text{UCP}_{\mathcal{H}}(X)$ induces a ucp $\phi^c : C(X) \rightarrow \mathcal{B}(\mathcal{H})$ by way of restriction:

$$\phi^c = \phi|_{C(X)}.$$

In particular, if $\varrho_{x_0} \in \text{UCP}_{\mathcal{H}}(X)$ is spectral, then $\varrho_{x_0}^c$ is in the character space of $C(X)$.

Definition 5.3. (The Γ^c -transform) For each $\nu \in \text{POVM}_{\mathcal{H}}(X)$ let $\Gamma^c\nu$ denote the ucp map $C(X) \rightarrow \mathcal{B}(\mathcal{H})$ defined by

$$\Gamma^c\nu = \phi_\nu^c.$$

Proposition 5.2. Γ^c is properly C^* -affine. That is,

$$\Gamma^c \left(\sum_{j=1}^m t_j^* \nu_j t_j \right) = \sum_{j=1}^m t_j^* (\Gamma^c \nu_j) t_j$$

for all $\nu_1, \dots, \nu_m \in \text{POVM}_{\mathcal{H}}(X)$ all invertible C^* -convex coefficients $t_1, \dots, t_m \in \mathcal{B}(\mathcal{H})$.

Proof. Suppose first that $\nu \in \text{POVM}_{\mathcal{H}}(X)$ and that $t \in \mathcal{B}(\mathcal{H})$ is invertible. We shall show that

$$(7) \quad \Gamma^c(t^* \nu t) = t^* (\Gamma^c \nu) t.$$

To this end, let $\mu = \frac{1}{d} \text{Tr} \circ \nu$, $\tilde{\nu} = \frac{1}{d} \text{Tr} \circ (t^* \nu t)$, and $\tilde{\mu} = \frac{1}{d} \text{Tr} \circ \tilde{\nu}$. Clearly $\tilde{\mu} \ll_{\text{ac}} \mu$ because $\tilde{\mu} \ll_{\text{ac}} \nu \ll_{\text{ac}} \mu$. Conversely, if $\tilde{\mu}(E) = 0$, then $t^* \nu(E) t = 0$ and so $\nu(E) = 0$ since t is invertible. Thus, $\mu \ll_{\text{ac}} \tilde{\mu}$.

For any $\psi \in C(X)$ and $\rho \in S(\mathcal{H})$, we have

$$\begin{aligned} \text{Tr} \left(\rho \int_X \psi d(t^* \nu t) \right) &= \int_X \psi \text{Tr} \left(\rho \frac{d(t^* \nu t)}{d\tilde{\mu}} \right) d\tilde{\mu} \\ &= \int_X \psi \text{Tr} \left(\rho t^* \frac{d\nu}{d\tilde{\mu}} t \right) d\tilde{\mu} \\ &= \int_X \psi \text{Tr} \left(\rho t^* \frac{d\nu}{d\mu} t \right) \frac{d\mu}{d\tilde{\mu}} d\tilde{\mu} \\ &= \int_X \psi \text{Tr} \left(\rho t^* \frac{d\nu}{d\mu} t \right) d\mu \\ &= \text{Tr} \left(\rho \left[t^* \left(\int_X \psi d\nu \right) t \right] \right). \end{aligned}$$

Hence, (7) holds.

Suppose next that $\nu_1, \nu_2 \in \text{POVM}_{\mathcal{H}}(X)$ and that $\lambda_1, \lambda_2 \in \mathbb{R}_+$ satisfy $\lambda_1 + \lambda_2 = 1$. Let $\nu = \lambda_1 \nu_1 + \lambda_2 \nu_2$, and let $\mu, \mu_1, \mu_2 \in P(X)$ denote the induced probability measures. Hence, $\mu_j \ll_{\text{ac}} \mu$ for $j = 1, 2$. Furthermore, for every $\psi \in C(X)$ and $\rho \in S(\mathcal{H})$ we have that

$$\begin{aligned} \int_X \psi \text{Tr} \left(\rho \frac{d\nu}{d\mu} \right) d\mu &= \int_X \psi \text{Tr} \left(\rho \left[\lambda_1 \frac{d\mu_1}{d\mu} \frac{d\nu_1}{d\mu_1} + \lambda_2 \frac{d\mu_2}{d\mu} \frac{d\nu_2}{d\mu_2} \right] \right) d\mu \\ &= \lambda_1 \int_X \psi \left[\text{Tr} \left(\rho \frac{d\nu_1}{d\mu_1} \right) \right] \frac{d\mu_1}{d\mu} d\mu + \lambda_2 \int_X \psi \left[\text{Tr} \left(\rho \frac{d\nu_2}{d\mu_2} \right) \right] \frac{d\mu_2}{d\mu} d\mu \\ &= \lambda_1 \int_X \psi \text{Tr} \left(\rho \frac{d\nu_1}{d\mu_1} \right) d\mu_1 + \lambda_2 \int_X \psi \text{Tr} \left(\rho \frac{d\nu_2}{d\mu_2} \right) d\mu_2. \end{aligned}$$

Thus, $\Gamma^c(\lambda_1 \nu_1 + \lambda_2 \nu_2) = \lambda_1 \Gamma^c \nu_1 + \lambda_2 \Gamma^c \nu_2$. This fact together with (7) implies that Γ^c is properly C^* -affine. \square

A ucp map $C(X) \rightarrow \mathcal{B}(\mathcal{H})$ has, in principle, many different ucp extensions to $C(X) \otimes \mathcal{B}(\mathcal{H})$. However, if a positive linear map $\varphi : C(X) \rightarrow \mathcal{B}(\mathcal{H})$ has the form

$$(8) \quad \varphi(g) = \sum_{j=1}^n g(x_j) b_j^* b_j$$

for some distinct $x_1, \dots, x_n \in X$ and C^* -convex coefficients $b_1, \dots, b_n \in \mathcal{B}(\mathcal{H})$, then the *natural extension* $\mathbf{e}(\varphi)$ of φ from $C(X)$ to $C(X) \otimes \mathcal{B}(\mathcal{H})$ is defined by

$$\mathbf{e}(\varphi) = \sum_{j=1}^n b_j^* \varphi_{x_j} b_j.$$

Observe that of $\varphi_1, \dots, \varphi_m : C(X) \rightarrow \mathcal{B}(\mathcal{H})$ are ucp maps of the form (8), then

$$(9) \quad \mathbf{e} \left(\sum_{j=1}^m t_j^* \varphi_j t_j \right) = \sum_{j=1}^m t_j^* \mathbf{e}(\varphi_j) t_j$$

for all invertible C^* -convex coefficients $t_1, \dots, t_m \in \mathcal{B}(\mathcal{H})$.

A further direct consequence of the definition: if $\nu \in \text{POVM}_{\mathcal{H}}(X)$ has finite support, then

$$(10) \quad \Omega \circ \epsilon \circ \Gamma^c \nu = \nu.$$

We now turn to the proof of Theorem 5.1.

Proof. Assume ν is a C*-extreme point of $\text{POVM}_{\mathcal{H}}(X)$. By Proposition 1.1, ν is necessarily an extremal POVM. Thus, by Theorem 4.1, there are distinct points $x_1, \dots, x_n \in X$ and operators $h_1, \dots, h_n \in \mathcal{B}(\mathcal{H})_+$ such that

$$\nu = \sum_{j=1}^n \delta_{x_j} h_j, \quad \phi_\nu = \sum_{j=1}^n h_j^{1/2} \varrho_{x_j} h_j^{1/2}, \quad \text{and} \quad \phi_\nu^c = \sum_{j=1}^n \varrho_{x_j}^c h_j.$$

Let $\mathcal{H}_\pi = \bigoplus_1^n \mathcal{H}$ and define $\pi : C(X) \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ by $\pi = \bigoplus_{j=1}^n \varrho_{x_j}^c$. Thus, $\varphi_\nu^c = w^* \pi w$, where $w : \mathcal{H} \rightarrow \mathcal{H}_\pi$ is the isometry $w\xi = \bigoplus h_j^{1/2} \xi$. Let \mathcal{M} be the commutant of $\pi(C(X))$. Because x_1, \dots, x_n are distinct, \mathcal{H}_π is generated by vectors of the form $\pi(\psi)w\xi$, for $\psi \in C(X)$ and $\xi \in \mathcal{H}$, and the commutant \mathcal{M} is given by $\mathcal{M} = \bigoplus_1^n \mathcal{B}(\mathcal{H})$.

Suppose that $\phi_\nu^c = \sum_{i=1}^m t_i^* \varphi_i t_i$ is a proper C*-convex combination of ucp maps $\varphi_i : C(X) \rightarrow \mathcal{B}(\mathcal{H})$. By the Radon-Nikodým theorem for completely positive maps [2, Theorem 1.4.2], for each i there is a positive contraction $a_i \in \mathcal{M}$ such that $t_i^* \varphi_i t_i = w^* a_i \pi w$. Because $a_i = \bigoplus_{j=1}^m a_j^i$ for some $a_j^i \in \mathcal{B}(\mathcal{H})_+$, we obtain

$$\varphi_i = \sum_{j=1}^n \varrho_{x_j}^c (b_j^i)^* b_j^i, \quad \text{where } b_j^i = (a_j^i)^{1/2} h_j^{1/2} t_i^{-1}.$$

Pass to the natural extension and apply Ω ; that is, let

$$\nu_i = \Omega(\epsilon(\varphi_i)) = \sum_{j=1}^n \delta_{x_j} (b_j^i)^* b_j^i.$$

Observe that $\Gamma^c \nu_i = \varphi_i$ and

$$\begin{aligned} \nu &= \Omega \circ \epsilon(\phi_\nu^c) = \Omega \circ \epsilon \left(\sum_{i=1}^m t_i^* \Gamma^c \nu_i t_i^* \right) \\ &= \Omega \circ \epsilon \circ \Gamma^c \left(\sum_{i=1}^m t_i^* \nu_i t_i^* \right) \\ &= \sum_{i=1}^m t_i^* \nu_i t_i^*. \end{aligned}$$

Because ν is a C*-extreme point, we obtain $\nu_i = u_i^* \nu u_i$ for some unitaries u_1, \dots, u_m in $\mathcal{B}(\mathcal{H})$. Now apply Γ^c to obtain $\varphi_i = u_i^* \phi_\nu u_i$ for each i . This proves that ϕ_ν^c is a C*-extreme point in the space of all ucp maps $C(X) \rightarrow \mathcal{B}(\mathcal{H})$. Because $C(X)$ is abelian and \mathcal{H} has finite dimension, all such extreme points are homomorphisms [14, Proposition 2.2], [16, Corollary 2.2]. It is readily verified that ϕ_ν^c is multiplicative if and only if each h_j is a projection and $h_j h_{j'} = h_j h_{j'} = 0$ for $j' \neq j$.

Conversely, suppose that ν is a projection-valued measure of the form $\nu = \sum_{j=1}^n \delta_{x_j} q_j$ for some distinct $x_1, \dots, x_n \in X$ and pairwise-orthogonal projections $q_1, \dots, q_n \in \mathcal{B}(\mathcal{H})$. Suppose that $\nu = \sum_{i=1}^m t_i^* \nu_i t_i$ is a proper C^* -convex combination. Because t_i is invertible, $\nu_i \ll_{ac} \nu$ (as we showed in the proof of Lemma 4.2), which implies that ν_i has finite support. Since Γ^c is properly affine, $\phi_\nu^c = \sum_{i=1}^m t_i^* \phi_{\nu_i}^c t_i$. But because ν is projection-valued, the ucp map $\phi_\nu^c : C(X) \rightarrow \mathcal{B}(\mathcal{H})$ is a homomorphism and, hence, C^* -extremal amongst all such ucp maps [14, Proposition 1.2]. Thus, there are unitaries $u_1, \dots, u_m \in \mathcal{B}(\mathcal{H})$ such that $\phi_{\nu_i}^c = u_i^* \phi_\nu u_i$ for all i . Hence,

$$\nu_i = \Omega \circ \epsilon \circ \Gamma^c \nu_i = \Omega \left(\sum_{j=1}^n u_i^* q_j \varrho_{x_j} q_j u_i \right) = u_i^* \nu u_i.$$

That is, ν is a C^* -extreme point of $\text{POVM}_\mathcal{H}(X)$. \square

5.1. Application: sharp measurements generate all quantum measurements through coarsening.

Theorem 5.3. *The C^* -convex hull of the C^* -extreme points of $\text{POVM}_\mathcal{H}(X)$ is dense in $\text{POVM}_\mathcal{H}(X)$. That is, every quantum measurement is approximated by coarsenings of sharp measurements.*

Proof. The C^* -convex hull of the C^* -extreme points of $\text{POVM}_\mathcal{H}(X)$ is, by Theorem 5.1, the set of all C^* -convex combinations of sharp measurements.

Select $\nu \in \text{POVM}_\mathcal{H}(X)$. By the Krein-Milman Theorem, there is a net $\{\nu_\alpha\}_\alpha \subset \text{POVM}_\mathcal{H}(X)$ such that each ν_α is a convex combination of extreme points of $\text{POVM}_\mathcal{H}(X)$ and $\nu_\alpha \rightarrow \nu$. Therefore, it is sufficient to show that every extreme point of $\text{POVM}_\mathcal{H}(X)$ is a coarsening of sharp measurements. Recall that Lemma 4.3 asserts that an extreme point ν' of $\text{POVM}_\mathcal{H}(X)$ must have finite support; that, is ν' has the form $\nu' = \sum_{j=1}^m \delta_{x_j} h_j$ for some $x_1, \dots, x_m \in X$ and $h_1, \dots, h_m \in \text{Eff}(\mathcal{H})$.

By way of the (sharp) scalar-valued quantum probability measures $E \mapsto \delta_{x_j}(E)1$ and the C^* -convex coefficients $a_j = h_j^{1/2}$ we obtain $\nu' = \sum_{j=1}^m a_j^* \delta_{x_j} a_j$, which is a C^* -convex combination of C^* -extreme points of $\text{POVM}_\mathcal{H}(X)$. \square

6. DISCUSSION

Our use of the terms “classical” and “nonclassical” in this paper inherently refer to “scalar valued” and “operator valued.” As noted by the referee, there is a highly nonclassical feature to what we are calling classical convexity. For example, a mixed state generally admits many distinct decompositions as a (classical) convex combination of pure states, a fact which is studied in great detail in [6] and lies at the heart of many of the difficulties in the interpretation of quantum mechanics [5].

Likewise, we use the term “quantum measurement” interchangeably with positive operator-valued probability measure. In this regard we are following a common (as in [3], for example) although not universal practice. A more refined terminology would use the term observable where we have used measurement, the term

instrument for the next level in which the accompanying state changes are taken into account, and reserve the term measurement for the highest level in which the entire description of the measurement model is given.

The integral representations afforded by the transforms Γ and Ω are related to Fujimoto's cp-convexity [17]. However, Fujimoto's cp-convexity is possibly too abstract to yield results as specific as those of the present paper. A more concrete yet still nonclassical notion of convexity is that of "matrix convexity" [13, 24], which is slightly more general than C^* -convexity. If for each $d \in \mathbb{N}$ one selects a d -dimensional Hilbert space \mathcal{H}_d , then one defines

$$\text{POVM}(X) = (\text{POVM}_{\mathcal{H}_d}(X))_{d \in \mathbb{N}},$$

which is a matrix convex set. One may adapt the transforms Γ and Ω to study $\text{POVM}(X)$ by way of unital completely positive linear maps $\psi : C(X) \otimes \mathcal{B}(\mathcal{H}_{d_1}) \rightarrow \mathcal{B}(\mathcal{H}_{d_2})$ for arbitrary $d_1, d_2 \in \mathbb{N}$. However, the Krein-Milman Theorem in matrix convexity [24] does not extend to C^* -convexity, making it necessary to establish Theorem 5.3 herein by direct methods.

The transform Γ^c is well known. In the setting of Hilbert space, a good discussion is in Davies's book [10]—indeed, Theorem 4.1.2 of [10] is especially relevant. A very general theory of Γ^c is achieved by Ylinen's work on regular transformation measures [25]. Ylinen has not restricted his study to (finite-dimensional) Hilbert space as we have done; he considers, more generally, arbitrary Banach and dual spaces in his framework.

To endow $\text{POVM}_{\mathcal{H}}(X)$ with a natural topology, we have opted to make use of the transforms Γ and Ω rather than, as is done in [20], using the Γ^c transform. Our main reason for this preference is because, in our view, Γ^c is a hybrid of classical and nonclassical notions, whereas Γ is purely nonclassical.

The concept of coarseness is an order relation on $\text{POVM}_{\mathcal{H}}(X)$ determined by quantum noise. In this regard, a fine measurement is maximal with respect to the order. There are other orders of interest, such as those related to cleanings [4] and smearings [21] of measurements and observables.

We have focused upon the case of compact X , but if X is locally compact but not compact, then one may consider the abelian C^* -algebra $C_0(X)$ of all continuous functions $X \rightarrow \mathbb{C}$ that vanish at infinity. Let \tilde{X} shall denote the one-point compactification of X . Thus, $C_0(\tilde{X}) = C(\tilde{X})$, the unital C^* -algebra of all continuous functions $\tilde{X} \rightarrow \mathbb{C}$. The C^* -algebra $C(\tilde{X}) \otimes \mathcal{B}(\mathcal{H})$ is isomorphic to the unitisation $(C_0(X) \otimes \mathcal{B}(\mathcal{H}))^\sim$ of the non-unital algebra $C_0(X) \otimes \mathcal{B}(\mathcal{H})$. If a linear map $\phi_0 : C_0(X) \otimes \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is contractive and completely positive, then there is a unital completely positive (ucp) linear map $\phi : C_0(\tilde{X}) \otimes \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ extending ϕ_0 . Conversely, every ucp map $\phi : C_0(\tilde{X}) \otimes \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ restricts to a contractive completely positive linear map $\phi_0 : C_0(X) \otimes \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$. The set

$$\left\{ \phi|_{C_0(X) \otimes \mathcal{B}(\mathcal{H})} \mid \phi \text{ is a ucp map } C_0(\tilde{X}) \otimes \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \right\}$$

plays the role of $\text{UCP}_{\mathcal{H}}(C(Y) \otimes \mathcal{B}(\mathcal{H}))$ for compact Hausdorff Y . Because the passage from X to \tilde{X} amounts to nothing more than adjoining a unit to a nonunital C^* -algebra, one can make slight reformulations of the results of the paper to cover the case of non-compact X .

Finally, because optimisation of the outcome statistics of apparatuses often amounts to minimising a real-valued concave function defined on the space $\text{Eff}(\mathcal{H})$

of quantum effects, it seems appropriate to mention here that there is a nonclassical analogue, using the integral under study in the present paper, of the classical inequality of Jensen for convex functions. For every selfadjoint $a \in \mathcal{B}(\mathcal{H})_{\text{sa}}$ with spectrum in an open interval $J \subset \mathbb{R}$, one may define a normal operator $\vartheta(a) \in \mathcal{B}(\mathcal{H})$, for a function $\vartheta : J \rightarrow \mathbb{C}$, by functional calculus. Coupled with the Löwner ordering of selfadjoint operators, one has the notion of operator convex function. If $J \subset \mathbb{R}$ is an open interval, then a function $\vartheta : J \rightarrow \mathbb{R}$ is *operator convex* if

$$\vartheta(\alpha a + (1 - \alpha)b) \leq \alpha\vartheta(a) + (1 - \alpha)\vartheta(b),$$

for all $\alpha \in [0, 1]$, all selfadjoint operators $a, b \in \mathcal{B}(\mathcal{H})_{\text{sa}}$ with spectrum in J , and all finite-dimensional Hilbert spaces \mathcal{H} .

Theorem 6.1. (Jensen's Inequality in POVMs [15]) *If $J \subset \mathbb{R}$ is an open interval containing a closed interval $[\alpha, \beta]$, and if $\kappa : X \rightarrow \mathcal{B}(\mathcal{H})$ is a Borel-measurable function for which $\kappa(x)$ is hermitian and has spectrum contained in $[\alpha, \beta]$ for every $x \in X$, then for any $\nu \in \text{POVM}_{\mathcal{H}}(X)$ we have*

$$\vartheta\left(\int_X \kappa d\nu\right) \leq \int_X \vartheta \circ \kappa d\nu,$$

for every operator convex function $\vartheta : J \rightarrow \mathbb{R}$.

In the case where X is a finite sample space, Theorem 6.1 is the Hansen–Pedersen–Jensen Inequality [18]: for any C^* -convex combination $\sum_{j=1}^m a_j^* y_j a_j$ of selfadjoint operators $y_1, \dots, y_m \in \mathcal{B}(\mathcal{H})$ with spectrum in an open interval J , and for any operator convex function $\vartheta : J \rightarrow \mathbb{R}$, the following operator inequality holds:

$$\vartheta\left(\sum_{j=1}^m a_j^* y_j a_j\right) \leq a_j^* \vartheta(y_j) a_j.$$

7. CONCLUSION

In this paper we have studied the structure of the set $\text{POVM}_{\mathcal{H}}(X)$ of quantum measurements of a quantum system (represented by a d -dimensional Hilbert space \mathcal{H}) whose possible measurement events is the σ -algebra $\mathcal{O}(X)$ of Borel sets of a compact Hausdorff space X . The classical case occurs with $d = 1$ and reduces to the study of probability measures. In classical analysis, one may integrate scalar-valued Borel functions with respect to arbitrary probability measures; so doing produces a positive linear functional on the abelian C^* -algebra $C(X)$. Herein we have defined an integral so that one may integrate any quantum random variable against an arbitrary positive operator-valued measure, and this has been achieved in a manner by which one produces a unital completely positive linear map of the homogeneous C^* -algebra $C(X) \otimes \mathcal{B}(\mathcal{H})$ into the I_d -factor $\mathcal{B}(\mathcal{H})$. Conversely, we have shown that there is a subclass of ucp maps $\phi : C(X) \otimes \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ such that each induces a positive operator-valued measure $\nu \in \text{POVM}_{\mathcal{H}}(X)$.

The transforms Γ and Ω allow one to move between $\text{POVM}_{\mathcal{H}}(X)$ and $\text{UCP}_{\mathcal{H}}(X)$. The transform Ω is C^* -affine, which is sufficient structure to topologise $\text{POVM}_{\mathcal{H}}(X)$ using the BW-topology of $\text{UCP}_{\mathcal{H}}(X)$ and to show that $\text{POVM}_{\mathcal{H}}(X)$ is a compact C^* -convex space. We have described precisely the structure of the extremal and C^* -extremal quantum measurements. The latter are precisely the sharp observables, while the former are certain positive operator-valued measures with finite support and which were determined for finite and arbitrary X by different methods in

some earlier works [7, 8, 9, 22]. As a consequence of the structure of extreme points and the Krein-Milman Theorem, every quantum measurement that one can perform in principle can be approximated by quantum measurements that one can perform in practice. That is, for every arbitrary quantum measurement ν (with perhaps infinitely many measurement outcomes) there is a quantum measurement ν' on a finite subspace $X' \subset X$ in which the measurement statistics of the subspace approximate those of the general measurement ν . By the nonclassical Krein-Milman Theorem (Theorem 5.3), the approximate ν' is a coarsening of a finite number of sharp measurements, each with measurement events $\mathcal{O}(X')$.

ACKNOWLEDGEMENT

We acknowledge the support of the NSERC Discovery, PGS, and USRA programs and Nipissing University (North Bay, Canada), where this work was undertaken during an extended scientific visit of the first author. We are especially indebted to Giulio Chiribella for drawing our attention to the works [7, 8], and to Michael Kozdron and the referee for useful commentary on the results herein.

REFERENCES

- [1] H. Araki and F. Hansen, *Jensen's operator inequality for functions of several variables*, Proc. Amer. Math. Soc. **128** (2000), no. 7, 2075–2084.
- [2] W. Arveson, *Subalgebras of C^* -algebras*, Acta Math. **123** (1969), 141–224.
- [3] I. Bengtsson and K. Życzkowski, *Geometry of quantum states*, Cambridge University Press, Cambridge, 2006.
- [4] F. Buscemi, M. Keyl, G. M. D'Ariano, P. Perinotti, and R. F. Werner, *Clean positive operator valued measures*, J. Math. Phys. **46** (2005), no. 8, 082109.
- [5] P. Busch, P.J. Lahti, and P. Mittelstaedt, *The quantum theory of measurement*, Lecture Notes in Physics m2, Springer-Verlag, Berlin, 1991.
- [6] G. Cassinelli, E. De Vito, and A. Leviero, *On the decompositions of a quantum state*, J. Math. Anal. Appl. **210** (1997), no. 2, 472–483.
- [7] G. Chiribella, G. M. D'Ariano, and D. Schlingemann, *How continuous quantum measurements in finite dimensions are actually discrete*, Phys. Rev. Lett. **98** (2007), no. 19, 190403.
- [8] G. Chiribella G. M. D'Ariano, and D. Schlingemann, *Barycentric decompositions of quantum measurements in finite dimensions*, J. Math. Phys. **51** (2010), no. 2, 0221111.
- [9] G. M. D'Ariano, P. Lo Presti, and P. Perinotti, *Classical randomness in quantum measurements*, J. Phys. A **38** (2005), no. 26, 5979–5991.
- [10] E. B. Davies, *Quantum theory of open systems*, Academic Press [Harcourt Brace Jovanovich Publishers], London, 1976.
- [11] J. Dixmier, *Les C^* -algèbres et leurs représentations*, Deuxième édition. Cahiers Scientifiques, Fasc. XXIX, Gauthier-Villars Éditeur, Paris, 1969.
- [12] R. G. Douglas, *On extremal measures and subspace density*, Michigan Math. J. **11** (1964), 243–246.
- [13] E. G. Effros, *A matrix convexity approach to some celebrated quantum inequalities*, Proc. Natl. Acad. Sci. USA **106** (2009), no. 4, 1006–1008.
- [14] D. R. Farenick and P. B. Morenz, *C^* -extreme points in the generalised state spaces of a C^* -algebra*, Trans. Amer. Math. Soc. **349** (1997), no. 5, 1725–1748.
- [15] D. R. Farenick and F. Zhou, *Jensen's inequality relative to matrix-valued measures*, J. Math. Anal. Appl. **327** (2007), no. 2, 919–929.
- [16] D. R. Farenick and H. Zhou, *The structure of C^* -extreme points in spaces of completely positive linear maps on C^* -algebras*, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1467–1477.
- [17] I. Fujimoto, *Decomposition of completely positive maps*, J. Operator Theory **32** (1994), no. 2, 273–297.
- [18] F. Hansen and G. K. Pedersen, *Jensen's operator inequality*, Bull. London Math. Soc. **35** (2003), no. 4, 553–564.

- [19] N. J. Higham, *Stable iterations for the matrix square root*, Numer. Algorithms **15** (1997), no. 2, 227–242.
- [20] A. S. Holevo, *Statistical structure of quantum theory*, Lecture Notes in Physics. Monographs, vol. 67, Springer-Verlag, Berlin, 2001.
- [21] A. Jenčová, S. Pulmannová, and E. Vinclová, *Sharp and fuzzy observables on effect algebras*, Internat. J. Theoret. Phys. **47** (2008), no. 1, 125–148.
- [22] K. R. Parthasarathy, *Extremal decision rules in quantum hypothesis testing*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **2** (1999), no. 4, 557–568.
- [23] V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, vol. 78, Cambridge University Press, Cambridge, 2002.
- [24] C. Webster and S. Winkler, *The Krein-Milman theorem in operator convexity*, Trans. Amer. Math. Soc. **351** (1999), no. 1, 307–322.
- [25] K. Ylinen, *Quantum instruments and related transformation valued functions*, Found. Phys. **39** (2009), no. 6, 656–675.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF REGINA, REGINA, SASKATCHEWAN S4S 0A2, CANADA

E-mail address: douglas.farenick@uregina.ca

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF GUELPH, GUELPH, ONTARIO N1G 2W1, CANADA

E-mail address: splosker@uoguelph.ca

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO, ONTARIO M5S 2E4, CANADA

E-mail address: smith36j@uregina.ca